GLUING RESTRICTED NERVES OF ∞-CATEGORIES

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Abstract. In this article, we develop a general technique for gluing subcategories of ∞-categories and prove that certain maps of simplicial sets are categorical equivalences. This applies in particular to maps naturally arising from the study of algebraic geometry, and allows us to upgrade the theory of derived categories of étale sheaves on schemes to the ∞-categorical level. In a subsequent article [13], we will use this technique to establish a general theory of Grothendieck's six operations for Artin stacks.

Contents

Introduction

In SGA 4 XVII [3], Deligne developed a theory of compactly-supported cohomology of schemes. More precisely, for a separated morphism $f\colon Y \to X$ of finite type between quasi-compact and quasi-separated schemes and a torsion ring Λ, he constructed an extraordinary pushforward functor

$$f_! \colon \mathrm{D}(Y,\Lambda) \to \mathrm{D}(X,\Lambda), \tag{0.1}$$

where $\mathrm{D}(-,\Lambda)$ denotes the unbounded derived category of étale Λ-modules. The functoriality of this operation is encoded by a pseudofunctor

$$F_! \colon \mathcal{S}\mathrm{ch}' \to \mathcal{C}\mathrm{at}_1 \tag{0.2}$$

sending a scheme X in $\mathcal{S}\mathrm{ch}'$ to $\mathrm{D}(X,\Lambda)$ and a morphism $f\colon Y\to X$ in $\mathcal{S}\mathrm{ch}'$ to the functor (0.1). Here $\mathcal{S}\mathrm{ch}'$ denotes the category of quasi-compact and quasi-separated schemes, with separated morphisms of finite type; and $\mathcal{C}\mathrm{at}_1$ denotes the $(2,1)$-category of categories[1]. The operation $f_!$, together with f^*, $\mathrm{R}f_*$, $f^!$, the derived tensor product, and the derived internal Hom, are known as Grothendieck's six operations for schemes in $\mathcal{S}\mathrm{ch}'$.

The main goal of the current article is to provide the necessary category-theoretic tool for a subsequent series of articles [13, 14], in which we develop a general theory of Grothendieck's six operations for lisse-étale sheaves on (higher) Artin stacks, extending the existing theories of [18] (for Deligne–Mumford stacks), [10], [1], [17] and [11, 12]. There we make essential use of the ∞-categorical descent in the context of Lurie's theory [15, 16] of ∞-categories, which are also known as quasi-categories. We refer the reader to the Introduction of [13] for a detailed explanation of our approach. As a starting point for the

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[1] A $(2,1)$-category is a 2-category in which all 2-cells are invertible.

descent procedure, one needs an ∞-categorical generalization of the pseudofunctor $F_!$ (0.2), namely, a functor

$$F_!^\infty\colon \mathrm{N}(\mathcal{S}\mathrm{ch}') \to \mathcal{C}\mathrm{at}_\infty \tag{0.3}$$

between ∞-categories, where $\mathrm{N}(\mathcal{S}\mathrm{ch}')$ is the nerve of $\mathcal{S}\mathrm{ch}'$ and $\mathcal{C}\mathrm{at}_\infty$ denotes the ∞-category of ∞-categories. For every scheme X in $\mathcal{S}\mathrm{ch}'$, $F_!^\infty(X)$ is an ∞-category $\mathcal{D}(X,\Lambda)$, whose homotopy category is $\mathrm{D}(X,\Lambda)$. For every morphism $f\colon Y\to X$ in $\mathcal{S}\mathrm{ch}'$, that is, a 1-cell of $\mathrm{N}(\mathcal{S}\mathrm{ch}')$, the image $F_\infty(f)$ is a functor

$$f_!^\infty\colon \mathcal{D}(Y,\Lambda)\to\mathcal{D}(X,\Lambda)$$

such that the induced functor $\mathrm{h}f_!^\infty$ between homotopy categories is equivalent to the classical $f_!$, whenever the latter is defined.

Deligne's construction of $F_!$ (0.2) relies on Nagata compactifications. Recall that, by [2], every morphism $f\colon Y\to X$ in $\mathcal{C}$ can be factorized in the following way:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "p"] \\ & Y, \end{tikzcd}$$

where i is an open immersion and p is proper. One can then define $f_!$ as $p_*\circ j_!$. The major issue is that such a factorization is not canonical. In order to construct the pseudofunctor $F_!$, one needs to make a coherent choice of factorizations. This is exactly what Deligne did. Since the target of $F_!$ is a $(2,1)$-category, he only needs to consider coherence up to 2-cells. The target of $F_!^\infty$ being an ∞-category, we need to consider coherence of *all* levels in our construction of $F_!^\infty$.

Another complication is the need to consider factorizations into more than two morphisms. As many important moduli stacks are not quasi-compact, in [13] we work with Artin stacks that are not necessarily quasi-compact. Accordingly, our actual starting point is a category $\mathcal{S}\mathrm{ch}''$ slightly bigger than $\mathcal{S}\mathrm{ch}'$, so that every Artin stack admits an atlas from an object of $\mathcal{S}\mathrm{ch}''$. The objects of $\mathcal{S}\mathrm{ch}''$ are disjoint unions of quasi-compact and quasi-separated schemes and the morphisms are separated morphisms *locally* of finite type. Every morphism f in $\mathcal{S}\mathrm{ch}''$ can be written as $f=q\circ p\circ j$, where q is a local isomorphism [7, 4.4.2], p is proper, and j is an open immersion. The correct definition of $f_!$ is $q_!\circ p_*\circ j_!$, which suggests that the factorization cannot be simplified.

These considerations lead us to propose the following general framework. Let $\mathcal{C}$ be an (ordinary) category and $k\geq 2$ be a finite number. Let $\mathcal{E}_1,\dots,\mathcal{E}_k\subseteq \mathrm{Ar}(\mathcal{C})$ be k sets of arrows of $\mathcal{C}$, each of which is stable under composition and contains all identity morphisms in $\mathcal{C}$. In addition to the nerve $\mathrm{N}(\mathcal{C})$ of $\mathcal{C}$, we define another simplicial set, which we denote by $\delta_k^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$. Its n-cells are functors $[n]^k\to\mathcal{C}$ such that the image of a morphism in the i-th direction is in $\mathcal{E}_i$ for $1\leq i\leq k$, and the image of every "unit square" is a Cartesian square. For example, when k=2, the n-cells of $\delta_2^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2}$ correspond to diagrams

$$\begin{tikzcd} c_{00} \arrow[r] \arrow[d] & c_{01} \arrow[r] \arrow[d] & \cdots \arrow[r] & c_{0n} \arrow[d] \\ c_{10} \arrow[r] \arrow[d] & c_{11} \arrow[r] \arrow[d] & \cdots \arrow[r] & c_{1n} \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\ c_{n0} \arrow[r] & c_{n1} \arrow[r] & \cdots \arrow[r] & c_{nn} \end{tikzcd} \tag{0.4}$$

where vertical (resp. horizontal) arrows are in $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) and all squares are Cartesian. The face and degeneration maps are defined in the obvious way. Note that $\delta_k^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$ is *seldom* an ∞-category. It is the simplicial set associated to a k-simplicial set $\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$. The latter is a special case of what we

call the *(restricted) multisimplicial nerve* of an (∞-)category (Definition 3.5). Let us mention in passing that this k-simplicial set can be interpreted as the k-fold nerve in the sense of Fiore and Paoli [4, 2.14] of a suitable k-fold category.

Let $\mathcal{E}_0 \subseteq \mathrm{Ar}(\mathcal{C})$ be a set of arrows stable under composition and containing $\mathcal{E}_1$ and $\mathcal{E}_2$. Then there is a natural map

$$g\colon \delta_k^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\dots,\mathcal{E}_k} \to \delta_{k-1}^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_0,\mathcal{E}_3,\dots,\mathcal{E}_k} \tag{0.5}$$

of simplicial sets, sending an n-cell of the source corresponding to a functor $[n]^k \to \mathcal{C}$, to its partial diagonal

$$[n]^{k-1} = [n] \times [n]^{k-2} \xrightarrow{\mathrm{diag}\times\mathrm{id}_{[n]^{k-2}}} [n]^k = [n]^2 \times [n]^{k-2} \to \mathcal{C},$$

which is an n-cell of the target.

A subset $\mathcal{E} \subseteq \mathrm{Ar}(\mathcal{C})$ is *admissible* (see Definition 3.3 for the general and precise definition) if $\mathcal{E}$ contains every identity morphism, is stable under pullback, and such that for every pair of composable maps $u \in \mathcal{E}$ and v in $\mathcal{C}$, $u \circ v$ is in $\mathcal{E}$ if and only if $v \in \mathcal{E}$.

Theorem 0.1 (Special case of Corollary 6.16 and Remark 6.17)**.** *Assuming the data* $(\mathcal{C}; \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_k)$ *satisfy the following assumptions:*

(1) $\mathcal{C}$ *admits pullbacks.*
(2) *For* $0 \le i \le k$, $\mathcal{E}_i$ *is stable under pullback (in particular,* $\mathcal{E}_i$ *contains all isomorphisms).*
(3) *For* $\alpha = 1, 2$, $\mathcal{E}_\alpha$ *is admissible and is contained in* $\mathcal{E}_0$.
(4) *Every morphism* f *of* $\mathcal{E}_0$ *is of the form* $p \circ q$, *where* $p \in \mathcal{E}_1$ *and* $q \in \mathcal{E}_2$.

Then the natural map (0.5)

$$g\colon \delta_k^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\dots,\mathcal{E}_k} \to \delta_{k-1}^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_0,\mathcal{E}_3,\dots,\mathcal{E}_k}$$

is a categorical equivalence [15, 1.1.5.14].

In the above discussion, we may replace $\mathrm{N}(\mathcal{C})$ by an ∞-category $\mathcal{C}$ (not necessarily the nerve of an ordinary category), and define $\delta_k^*\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$. Moreover, in [13], we need to encode information besides the !-pushforward such as the Base Change isomorphism, which involves both $*$-pullback and !-pushforward. To this end, we will define in Section 3, for every subset $L \subseteq \{1, \dots, k\}$, a variant $\delta_{k,L}^*\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$ of $\delta_k^*\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$ by "taking the opposite" in the directions in L. For $L \subseteq \{3, \dots, k\}$, the theorem remains valid modulo slight modifications. We refer the reader to Corollary 6.16 for a precise statement.

Corollary 0.2. *Let* $P \subseteq \mathrm{Ar}(\mathcal{S}\mathrm{ch}')$ *be the subset of proper morphisms,* $J \subseteq \mathrm{Ar}(\mathcal{S}\mathrm{ch}')$ *be the subset of open immersions. Then the natural map*

$$\delta_2^*\mathrm{N}(\mathcal{S}\mathrm{ch}')^{\mathrm{cart}}_{P,J} \to \mathrm{N}(\mathcal{S}\mathrm{ch}')$$

is a categorical equivalence.

Proof. This follows immediately from the theorem with $k = 2$, $\mathcal{C} = \mathcal{S}\mathrm{ch}'$, $\mathcal{E}_0 = \mathrm{Ar}(\mathcal{S}\mathrm{ch}')$, $\mathcal{E}_1 = P$, $\mathcal{E}_2 = J$. □

Corollary 0.3. *Let* $F = \mathrm{Ar}(\mathcal{S}\mathrm{ch}'')$, $P \subseteq F$ *be the subset of proper morphisms,* $I \subseteq F$ *be the subset of local isomorphisms. Then the natural map*

$$\delta_2^*\mathrm{N}(\mathcal{S}\mathrm{ch}'')^{\mathrm{cart}}_{P,I} \to \mathrm{N}(\mathcal{S}\mathrm{ch}'')$$

is a categorical equivalence.

The corollary still holds if one replaces I by the subset $E \subseteq F$ of étale morphisms.

One might be tempted to apply Theorem 0.1 directly by taking $k = 2$, $\mathcal{E}_0 = F$, $\mathcal{E}_1 = P$, $\mathcal{E}_2 = E$. However, assumption (4) of the theorem is not satisfied.

Proof of Corollary 0.3. Let $\mathcal{C} = \mathcal{S}\mathrm{ch}''$. Instead of applying the theorem directly, we introduce the following sets of morphisms. Let $F_{\mathrm{ft}} \subseteq F$ be the set of separated morphisms of finite type, $I_{\mathrm{ft}} = I \cap F_{\mathrm{ft}}$.

Consider the following commutative diagram

$$\begin{tikzcd}
\delta_3^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{P,I_{\mathrm{ft}},I} \arrow[r] \arrow[d] & \delta_2^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{F_{\mathrm{ft}},I} \arrow[d] \\
\delta_2^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{P,I} \arrow[r] & \mathrm{N}(\mathcal{C}),
\end{tikzcd}$$

where the upper arrow is induced by "composing morphisms in P and I_{ft}", while the left arrow is induced by "composing morphisms in I_{ft} and I". We apply Theorem 0.1 three times to show that all arrows except the lower one are categorical equivalences. Therefore, the lower arrow is also a categorical equivalence.

We will provide the details of the proof for the upper arrow and leave the other two to the reader. In the theorem, we let $k = 3$, $\mathcal{E}_0 = F_{\mathrm{ft}}$, $\mathcal{E}_1 = P$, $\mathcal{E}_2 = I_{\mathrm{ft}}$ and $\mathcal{E}_3 = I$. Assumptions (1), (2) and (3) are obviously satisfied, and (4) is satisfied by Nagata compactifications. $\square$

In some situations, it is possible to encode the information of a restricted bisimplicial nerve using an ∞-category, instead of a (bi)simplicial set. In fact, given an ∞-category $\mathcal{C}$ and classes of edges $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{C}_1$ stable under composition and containing all degenerate edges, Gaitsgory defined an ∞-category of correspondences $\mathcal{C}_{\mathrm{corr}:\mathcal{E}_1;\mathcal{E}_2}$ [5, 5.1.2] ($\mathbf{C}_{\mathrm{corr}:\alpha;\beta}$ in his notations, as he writes $\mathbf{C}$ for $\mathcal{C}$, α for $\mathcal{E}_1$, and β for $\mathcal{E}_2$). There is a natural map

$$f\colon \delta^*_{2,\{2\}}\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2} \to \mathcal{C}_{\mathrm{corr}:\mathcal{E}_1;\mathcal{E}_2}$$

given by "forgetting the lower-right corner of the cells", where $\delta^*_{2,\{2\}}\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2}$ is a variant of $\delta_2^*\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2}$ already mentioned in the remark following Theorem 0.1. It turns out, by Corollary 4.5, that f is a categorical equivalence. Despite this fact, we will not adopt the language of correspondences as it cannot replace simplicial sets such as $\delta_2^*\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2}$.

The proof of Theorem 0.1 consists of two steps. Let us illustrate them in the case where $k = 2$. In this case, we may assume that $\mathcal{E}_0 = \mathrm{Ar}(\mathcal{C})$ is the set of all morphisms. Then the theorem claims that the natural map

$$g\colon \delta_2^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2} \to \mathrm{N}(\mathcal{C})$$

is a categorical equivalence. The map g can be decomposed as

$$\delta_2^*\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2} \xrightarrow{g'} \delta_2^*\mathrm{N}(\mathcal{C})_{\mathcal{E}_1,\mathcal{E}_2} \xrightarrow{g''} \mathrm{N}(\mathcal{C}),$$

where $\delta_2^*\mathrm{N}(\mathcal{C})_{\mathcal{E}_1,\mathcal{E}_2}$ is the simplicial set whose n-cells are diagrams (0.4) without the requirement that every square is Cartesian, g' is the natural inclusion and g'' is the morphism remembering the diagonal. We will prove that both g' and g'' are categorical equivalences.

In Section 1, we recall some basic notions in the theory of ∞-categories [15] for the reader's convenience. In Section 2, we prove a key lemma (Lemma 2.2), which provides a general method for the construction of functors to ∞-categories. The lemma will be used several times in this article and its sequels [13, 14]. In Section 3, we introduce the notion of multisimplicial sets and several variants. In particular, we define the restricted multisimplicial nerve of an ∞-category. Although multisimplicial sets can be avoided in the statement of Theorem 0.1, they are quite natural and are necessary in the course of the proof. In Section 4, we prove that the map g'' is a categorical equivalence, by generalizing Deligne's idea. In order to prove that g' is a categorical equivalence as well, some combinatorics of finite lattices are needed. We record them in Section 5. Section 6 is the technical heart of the article, where we "glue" smaller restricted nerves to larger ones, and prove that g' is a categorical equivalence. On the level of 2-categories, a naive variant of this has already appeared in [19]. In Section 7, we prove that certain inclusions of simplicial sets are inner anodyne, which is used in the previous sections.

Conventions. Unless otherwise specified, a category is to be understood as an ordinary category. We will not distinguish between a set and a category in which the only morphisms are identity morphisms. Let $\mathcal{C}, \mathcal{D}$ be two categories. We denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the *category of functors* from $\mathcal{C}$ to $\mathcal{D}$, whose objects are functors and whose morphisms are natural transformations.

The word *∞-category* refers to the notion defined in [15, 1.1.2.4], which is called *quasi-category* in [8, 9]. Throughout this article, an effort has been made to keep our notations consistent with those in [15].

For $\mathcal{C}$ a category, or an ∞-category, we denote by $\mathrm{id}_{\mathcal{C}}$ the identity functor of $\mathcal{C}$.

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1. Simplicial sets and ∞-categories

In this section, we collect some basic definitions in Lurie's theory of ∞-categories [15]. For a more systematic introduction to Lurie's theory, we recommend [6].

For $n \geq 0$, we denote by $[n]$ the totally ordered set $\{0, \dots, n\}$ and we put $[-1] = \emptyset$. Let us recall the *category of combinatorial simplices*, $\mathbf{\Delta}$. Its objects are the linearly ordered sets $[n]$ for $n \geq 0$ and its morphisms are given by (nonstrictly) order-preserving maps. For every $n \geq 0$ and $0 \leq k \leq n$, the face map $d^n_k \colon [n-1] \to [n]$ is the unique injective order-preserving map such that k is not in the image; and the degeneration map $s^n_k \colon [n+1] \to [n]$ is the unique surjective order-preserving map such that $s^n_k(k+1) = s^n_k(k)$.

Definition 1.1 (Simplicial set and ∞-category). Let $\mathcal{S}\mathrm{et}$ be the category of sets[2].

- A *simplicial set* is an object of the category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{S}\mathrm{et})$. A map of simplicial sets is a morphism in this category. For a simplicial set S, we denote by $S_n = S([n])$ its set of n-cells.
- For $n \geq 0$, we denote by $\Delta^n = \mathrm{Fun}(-, [n])$ the simplicial set represented by $[n]$. For each $0 \leq k \leq n$, we define the *k-th horn* $\Lambda^n_k \subseteq \Delta^n$ to be the simplicial subset obtained by removing the interior and the face opposite to the k-th vertex. Here by the k-th vertex we mean the simplicial subset represented by $\{k\} \subseteq [n]$.
- An *∞-category* is a simplicial set $\mathcal{C}$ such that $\mathcal{C} \to *$ has the right lifting property with respect to all inclusions $\Lambda^n_k \subseteq \mathbf{\Delta}^n$ with $0 < k < n$. In other words, a simplicial set $\mathcal{C}$ is an ∞-category if and only if every map $\Lambda^n_k \to \mathcal{C}$ with $0 < k < n$ can be extended to a map $\Delta^n \to \mathcal{C}$. For a pair of objects (namely, 0-cells) X and Y of $\mathcal{C}$, one defines the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$, which is an object of $\mathcal{H}$, the homotopy category of spaces.
- Given a simplicial set S and an ∞-category $\mathcal{C}$, we define a simplicial set $\mathrm{Fun}(S, \mathcal{C})$ by $\mathrm{Fun}(S, \mathcal{C})_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(S \times \mathbf{\Delta}^n, \mathcal{C})$. It is not difficult to see that $\mathrm{Fun}(S, \mathcal{C})$ is an ∞-category.

Example 1.2. Let $\mathcal{C}$ be an (ordinary) category. The *nerve* $\mathrm{N}(\mathcal{C})$ of $\mathcal{C}$ is the simplicial set given by $\mathrm{N}(\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$. It is easy to see that $\mathrm{N}(\mathcal{C})$ is an ∞-category and we can identify $\mathrm{N}(\mathcal{C})_0$ and $\mathrm{N}(\mathcal{C})_1$ with the set of objects $\mathrm{Ob}(\mathcal{C})$ and the set of arrows $\mathrm{Ar}(\mathcal{C})$, respectively. Conversely, from an ∞-category $\mathcal{C}$, one constructs an (ordinary) category $\mathrm{h}\mathcal{C}$, the *homotopy category of* $\mathcal{C}$ (ignoring the $\mathcal{H}$-enrichment), such that $\mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathcal{C}_0$ and $\mathrm{hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y)$ consists of homotopy classes in $\mathcal{C}_1$. By [15, 1.2.3.1], h is a left adjoint to the nerve functor N.

The lifting property defining ∞-category can be adapted to the relative case. More precisely, a map $f \colon T \to S$ of simplicial sets is called an *inner fibration* if it has the right lifting property with respect to all inclusions $\Lambda^n_k \subseteq \mathbf{\Delta}^n$ with $0 < k < n$. A map $i \colon A \to B$ of simplicial sets is *inner anodyne* if it has the left lifting property with respect to all inner fibrations.

We now recall the notion of categorical equivalence of simplicial sets, which is essential to our article. There are several equivalent definitions of categorical equivalence. The one given below (equivalent to [15, 1.1.5.14] in view of [15, 2.2.5.8]), due to Joyal [9], will be used in the proofs of our theorems.

Definition 1.3 (Categorical equivalence). A map $f \colon T \to S$ of simplicial sets is a *categorical equivalence* if for every ∞-category $\mathcal{C}$, the induced functor

$$\mathrm{hFun}(S, \mathcal{C}) \to \mathrm{hFun}(T, \mathcal{C})$$

[2] More rigorously, $\mathcal{S}\mathrm{et}$ is the category of sets in a universe that we fix once and for all.

is an equivalence of (ordinary) categories.

In Section 3, we will introduce the notion of multi-marked simplicial sets, which generalizes the notion of marked simplicial sets in [15, 3.1.0.1]. Since marked simplicial sets play an important role in many arguments for ∞-categories, we briefly recall its definition.

A *marked simplicial set* is a pair $(X, \mathcal{E})$ where X is a simplicial set and $\mathcal{E} \subseteq X_1$ that contains all degenerate edges. A morphism $f\colon (X,\mathcal{E}) \to (X',\mathcal{E}')$ of marked simplicial set is a map $f\colon X \to X'$ of simplicial sets such that $f(\mathcal{E}) \subseteq \mathcal{E}'$. The category of marked simplicial set will be denoted by $\mathcal{S}\mathrm{et}_\Delta^+$. For example, for every simplicial set S, we have two marked simplicial sets $S^\flat = (S, \mathcal{E})$ where $\mathcal{E}$ is the set of all degenerate edges, and $S^\sharp = (S, S_1)$. In the following sections, we will use the *Cartesian model structure* on the category $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$ of marked simplicial sets over $S^\sharp$ constructed in [15, 3.1.3].

Let us record the following simple but very useful lemma.

Lemma 1.4. *Let K be a simplicial set and $\mathcal{C}$ be an ∞-category, $f, g\colon K \to \mathcal{C}$ be two functors and $\phi\colon f \to g$ be a natural transformation, that is, a 1-cell in* $\mathrm{Fun}(K,\mathcal{C})$ [15, 1.2.7.2]. *Then ϕ is a natural equivalence* [15, 1.2.7.2] *if and only if for every vertex k of K, the map $\phi(k)\colon f(k) \to g(k)$ is an equivalence* [15, 1.2.4].

Proof. The necessity is trivial. To prove the sufficiency, let $\mathcal{C}^\natural$ be the marked simplicial set $(\mathcal{C},\mathcal{E})$ [15, 3.1.0.1], where $\mathcal{E}$ is the set of all edges of $\mathcal{C}$ that are equivalences. By assumption, ϕ is a 1-cell in $\mathrm{Map}^\sharp(K^\flat, \mathcal{C}^\natural) \subseteq \mathrm{Fun}(K,\mathcal{C})$ [15, 3.1.3]. By [15, 3.1.3.6], $\mathrm{Map}^\sharp(K^\flat,\mathcal{C}^\natural)$ is a Kan complex [15, 1.1.2.1]. It follows that ϕ is an equivalence in $\mathrm{Fun}(K,\mathcal{C})$. □

2. The key lemma

In this section, we develop a general technique for the construction of functors (Lemma 2.2). This lemma is the key to many constructions in this article and its sequels.

Let T be a simplicial set. The Cartesian monoidal structure on $(\mathcal{S}\mathrm{et}_\Delta^+)_{/T}$ is right closed. For objects X and Y in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/T}$, we define

$$\mathrm{Map}_T^\natural(X,Y) := (\mathrm{Map}_T^\flat(X,Y), \mathrm{Map}_T^\sharp(X,Y)_1)$$

in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/T}$. For a map of simplicial sets $T \to S$ and A in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$, we have an isomorphism

$$\mathrm{Map}_S^\natural(A, \mathrm{Map}_T^\natural(X,Y)) \simeq \mathrm{Map}_T^\natural(A \times_S X, Y)$$

in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$. If $Z \to T$ is a Cartesian fibration [15, 2.4.2.1], $\mathrm{Map}_T^\natural(X, Z^\natural) = \mathrm{Map}_T^\flat(X,Z^\natural)^\natural$.

Lemma 2.1. *Let T be a simplicial set, $X \to Y$ be a fibration in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/T}$ with respect to the Cartesian model structure, $i\colon A \to B$ be a cofibration in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/T}$ with respect to the Cartesian model structure. Then the induced map*

$$\mathrm{Map}_T^\natural(B,X) \to \mathrm{Map}_T^\natural(A,X) \times_{\mathrm{Map}_T^\natural(A,Y)} \mathrm{Map}_T^\natural(B,Y)$$

is a fibration in $\mathcal{S}\mathrm{et}_\Delta^+$ with respect to the Cartesian model structure. In particular, the map

$$\mathrm{Map}_T^\sharp(B,X) \to \mathrm{Map}_T^\sharp(A,X) \times_{\mathrm{Map}_T^\sharp(A,Y)} \mathrm{Map}_T^\sharp(B,Y)$$

is a Kan fibration.

Proof. The first assertion follows from the fact that smash product by i sends trivial cofibrations in $\mathcal{S}\mathrm{et}_\Delta^+$ to trivial cofibrations in $(\mathcal{S}\mathrm{et}_\Delta^+)_{/T}$ [15, 3.1.4.3]. In the notations of Definition 3.2, $(\delta_{1+}^*, \delta_*^{1+})$ is a Quillen adjunction for the Kan model structure on $\mathcal{S}\mathrm{et}_\Delta$ and the Cartesian model structure on $\mathcal{S}\mathrm{et}_\Delta^+$. In fact, the functor $\delta_{1+}^*\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^+$ sending Z to $Z^\sharp$ preserves cofibrations and trivial cofibrations. Applying δ_*^{1+} to the first assertion, we obtain the second one. □

Let K be a simplicial set. Recall that the *category of simplices of* K [15, 6.1.2.5], denoted by $\mathbf{\Delta}_{/K}$, is the strict fiber product $\mathbf{\Delta} \times_{\mathcal{S}\mathrm{et}_\Delta} (\mathcal{S}\mathrm{et}_\Delta)_{/K}$. An object of $\mathbf{\Delta}_{/K}$ is a pair (J,σ), where $J \in \Delta$ and

$\sigma \in \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^J, K)$. A morphism $(J,\sigma) \to (J',\sigma')$ is a map $d\colon \Delta^J \to \Delta^{J'}$ such that $\sigma = \sigma' \circ d$. Let M be a marked simplicial set. We define an object $\mathrm{Map}[K,M]$ of the diagram category $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ by

$$\mathrm{Map}[K,M](J,\sigma) = \mathrm{Map}^\sharp((\Delta^J)^\flat, M),$$

for every $(J,\sigma) \in \mathbf{\Delta}_{/K}$. If $M = \mathcal{C}^\natural$ for some ∞-category $\mathcal{C}$, we let $\mathrm{Map}[K,\mathcal{C}] = \mathrm{Map}[K,M]$. Then $\mathrm{Map}[K,\mathcal{C}](J,\sigma)$ is the largest Kan complex [15, 1.2.5.3] contained in $\mathrm{Fun}(\Delta^J,\mathcal{C})$. A morphism d in $\mathbf{\Delta}_{/K}$ goes to the natural restriction map Res^d of Kan mapping complexes. The right adjoint of the diagonal functor $\mathcal{S}\mathrm{et}_\Delta \to (\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ is the global section functor

$$\Gamma\colon (\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}} \to \mathcal{S}\mathrm{et}_\Delta, \quad \Gamma(\mathcal{N})_q = \mathrm{Hom}_{(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}}(\Delta^q_K, \mathcal{N}),$$

where $\Delta^q_K\colon (\mathbf{\Delta}_{/K})^{op} \to \mathcal{S}\mathrm{et}_\Delta$ is the constant functor of value Δ^q. We have

$$\Gamma(\mathrm{Map}[K,\mathcal{C}]) = \mathrm{Map}^\sharp(K^\flat, \mathcal{C}^\natural).$$

If $g\colon K' \to K$ is a map, composition with the functor $\mathbf{\Delta}_{/K'} \to \mathbf{\Delta}_{/K}$ induced by g defines a functor $g^*\colon (\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}} \to (\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K'})^{op}}$. We have $g^*\mathrm{Map}[K,M] = \mathrm{Map}[K',M]$.

Let $\Phi\colon \mathcal{M} \to \mathcal{N}$ be a morphism of $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$. We denote by $\Gamma_\Phi(\mathcal{N}) \subseteq \Gamma(\mathcal{N})$ the simplicial subset which is the union of the image of $\Gamma(\Phi')\colon \Gamma(\mathcal{M}') \to \Gamma(\mathcal{N})$ for all decompositions

$$\mathcal{M} \hookrightarrow \mathcal{M}' \xrightarrow{\Phi'} \mathcal{N}$$

of Φ such that $\mathcal{M}(\sigma) \hookrightarrow \mathcal{M}'(\sigma)$ is anodyne [15, 2.0.0.3] for all $\sigma \in \mathbf{\Delta}_{/K}$. The map $\Gamma(\Phi)\colon \Gamma(\mathcal{M}) \to \Gamma(\mathcal{N})$ factors through $\Gamma_\Phi(\mathcal{N})$. For every map $g\colon K' \to K$, the canonical map $\Gamma(\mathcal{N}) \to \Gamma(g^*\mathcal{N})$ sends $\Gamma_\Phi(\mathcal{N})$ to $\Gamma_{g^*\Phi}(g^*\mathcal{N})$. For every morphism $\psi\colon \mathcal{N} \to \mathcal{N}'$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$, the map $\Gamma(\psi)\colon \Gamma(\mathcal{N}) \to \Gamma(\mathcal{N}')$ induces a map

$$\Gamma_\Phi(\psi)\colon \Gamma_\Phi(\mathcal{N}) \to \Gamma_{\psi\circ\Phi}(\mathcal{N}').$$

For every element $a \in \Gamma(\mathcal{N}')_0$ (resp. $a \in \Gamma_{\psi\circ\Phi}(\mathcal{N}')_0$), we denote by $\Gamma^\psi_a(\mathcal{N})$ (resp. $\Gamma^\psi_{\Phi,a}(\mathcal{N})$) the fiber of $\Gamma(\psi)$ (resp. $\Gamma_\Phi(\psi)$) at a. We omit ψ from the notation when no confusion arises.

We can now state and prove the key lemma.

Lemma 2.2. *Let $f\colon Z \to T$ be a fibration in $\mathcal{S}\mathrm{et}^+_\Delta$ with respect to the Cartesian model structure, K be a simplicial set, $a\colon K^\flat \to T$ be a map, $\mathcal{N} \in (\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ be such that $\mathcal{N}(\sigma)$ is weakly contractible for all $\sigma \in \mathbf{\Delta}_{/K}$. Consider morphisms $\Phi\colon \mathcal{N} \to \mathrm{Map}[K,Z]$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ such that $\mathrm{Map}[K,f]\circ\Phi$ factors through $a\colon \Delta^0_K \to \mathrm{Map}[K,T]$.*

1. *For every such Φ, $\Gamma_{\Phi,a}(\mathrm{Map}[K,Z])$ is a weakly contractible Kan complex.*
2. *Let $\Phi,\Phi'\colon \mathcal{N} \to \mathrm{Map}[K,Z]$ be two such morphisms. Assume that Φ and Φ' are homotopic with respect to a. Then $\Gamma_{\Phi,a}(\mathrm{Map}[K,Z])$ and $\Gamma_{\Phi',a}(\mathrm{Map}[K,Z])$ lie in the same connected component of $\Gamma_a(\mathrm{Map}[K,Z])$.*

Here, by saying Φ and Φ' are homotopic with respect to a, we mean that there exists a morphism $H\colon \Delta^1_K \times \mathcal{N} \to \mathrm{Map}[K,Z]$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ such that $H \mid \Delta^{\{0\}}_K \times \mathcal{N} = \Phi$, $H \mid \Delta^{\{1\}}_K \times \mathcal{N} = \Phi'$ and $\mathrm{Map}[K,f]\circ H$ is the identity on a.

By Lemma 2.1, $\Gamma(\mathrm{Map}[K,f])$ is a Kan fibration so that $\Gamma_a(\mathrm{Map}[K,Z])$ is a Kan complex. We will often apply the lemma to the fibration in Lemma 2.1.

Proof. (1) The simplicial set $\Gamma_{\Phi,a}(\mathrm{Map}[K,Z])$ is the colimit of the image of $\Gamma(\Psi)$, indexed by the filtered category $\mathcal{C}$ of triples $(\mathcal{M}, i, \Psi)$ fitting into commutative diagrams in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ of the form

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "\Phi"] \arrow[d, hook, "i"'] & & \mathrm{Map}[K,Z] \arrow[d, "\mathrm{Map}[K,f]"] \\
\mathcal{M} \arrow[r] \arrow[urr, "\Psi"] & \Delta^K_0 \arrow[r, "a"] & \mathrm{Map}[K,T]
\end{tikzcd}$$

such that $\mathcal{M}(\sigma)$ is weakly contractible for all objects σ in $\mathbf{\Delta}_{/K}$. A morphism $(\mathcal{M}, i, \Psi) \to (\mathcal{M}', i', \Psi')$ in $\mathcal{C}$ is a monomorphism $j\colon \mathcal{M} \to \mathcal{M}'$ such that $j \circ i = i'$ and $\psi = \psi' \circ j$. Consider the following lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \Gamma_{\Phi,a}(\mathrm{Map}[K,Z]) \\ \Delta^n. \arrow[ur, dotted] & \end{tikzcd}$$

The upper horizontal arrow corresponds to a morphism $(\partial\Delta^n)_K \to \mathcal{M}$, where $(\mathcal{M}, i, \Psi)$ is an object of $\mathcal{C}$. We define an object $\mathcal{M}' = (\mathcal{M} \coprod_{\partial\Delta^n_K} \Delta^n_K)^\triangleright$ of $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ by $\mathcal{M}'(\sigma) = (\mathcal{M}(\sigma) \coprod_{\partial\Delta^n} \Delta^n)^\triangleright$ for all objects σ of $\mathbf{\Delta}_{/K}$ and $\mathcal{M}'(d) = (\mathcal{M}(d) \coprod_{\partial\Delta^n} \Delta^n)^\triangleright$ for all morphisms d of $\mathbf{\Delta}_{/K}$. Applying Lemma 2.3 to the inclusion $\mathcal{M} \to \mathcal{M}'$, we obtain a morphism $(\mathcal{M}, i, \Psi) \to (\mathcal{M}', i', \Psi')$ in $\mathcal{C}$. Then the map $\Delta^n_K \to \mathcal{M}'$ provides the dotted arrow.

(2) We apply Lemma 2.3 to the square

$$\begin{tikzcd} \Delta^1_K \times \mathcal{N} \arrow[rr, "H"] \arrow[d, hook] & & \mathrm{Map}[K,Z] \arrow[d, "{\mathrm{Map}[K,f]}"] \\ \Delta^1_K \times \mathcal{N}^\triangleright \arrow[r] \arrow[urr, dotted, "H'"] & \Delta^0_K \arrow[r, "a"] & \mathrm{Map}[K,T] \end{tikzcd}$$

We denote by $h\colon \Delta^1_K \to \mathrm{Map}[K,Z]$ the restriction of H' to the cone point of $\mathcal{N}^\triangleright$. Then $h(0)$ belongs to $\Gamma_{\Phi,a}(\mathrm{Map}[K,Z])$ and $h(1)$ belongs to $\Gamma_{\Phi',a}(\mathrm{Map}[K,Z])$.

□

Lemma 2.3. *Let $f\colon Z \to T$ be a fibration in $\mathcal{S}\mathrm{et}^+_\Delta$ with respect to the Cartesian model structure, K be a simplicial set. For every commutative square in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ of the form*

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\Phi"] \arrow[d, hook] & \mathrm{Map}[K,Z] \arrow[d, "{\mathrm{Map}[K,f]}"] \\ \mathcal{M} \arrow[r, "\Psi"'] \arrow[ur, dotted, "\Phi'"] & \mathrm{Map}[K,T] \end{tikzcd}$$

such that $\mathcal{N}(\sigma) \hookrightarrow \mathcal{M}(\sigma)$ is anodyne for all $\sigma \in \mathbf{\Delta}_{/K}$, there exists a dotted arrow as indicated, rendering the diagram commutative.

Proof. We proceed by induction. Choose an exhaustion of K by a transfinite sequence of simplicial subsets

$$\emptyset = K^0 \subseteq K^1 \subseteq \cdots$$

where each K^α, $\alpha > 0$ is obtained from

$$K^{<\alpha} = \bigcup_{\beta<\alpha} K^\beta$$

by adjoining a single nondegenerate simplex σ^α, provided that such a simplex exists. Let $\mathcal{M}^{<\alpha} = (i^{<\alpha})^*\mathcal{M}$, $\mathcal{M}^\alpha = (i^\alpha)^*\mathcal{M}$, where $i^{<\alpha}\colon K^{<\alpha} \hookrightarrow K$ and $i^\alpha\colon K^\alpha \hookrightarrow K$ are the inclusions. Similarly, we put $\mathcal{N}^{<\alpha} = (i^{<\alpha})^*\mathcal{N}$, $\Phi^{<\alpha} = (i^{<\alpha})^*\Phi$, and $\Psi^{<\alpha} = (i^{<\alpha})^*\Psi$. Suppose that we have found the map $(\Phi')^{<\alpha}$, rendering the diagram

$$\begin{tikzcd} \mathcal{N}^{<\alpha} \arrow[r, "\Phi^{<\alpha}"] \arrow[d] & \mathrm{Map}[K^{<\alpha},Z] \arrow[d, "{\mathrm{Map}[K^{<\alpha},f]}"] \\ \mathcal{M}^{<\alpha} \arrow[r, "\Psi^{<\alpha}"'] \arrow[ur, "(\Phi')^{<\alpha}"] & \mathrm{Map}[K^{<\alpha},T] \end{tikzcd}$$

commutative. We are going to construct $(\Phi')^\alpha$. If σ^α does not exist, we take $(\Phi')^\alpha = (\Phi')^{<\alpha}$. If $\sigma^\alpha\colon \Delta^{J^\alpha} \to K$ exists, it amounts to construct a map $\Phi'(\sigma^\alpha)$ as the dotted arrow rendering the following diagram commutative

$$\begin{tikzcd} \mathcal{N}(\sigma^\alpha) \arrow[r, "\Phi(\sigma^\alpha)"] \arrow[d, hook] & \mathrm{Map}^\sharp((\Delta^{J^\alpha})^\flat, Z) \arrow[d, "\mathrm{Map}^\sharp((\Delta^{J^\alpha})^\flat, f)"] \\ \mathcal{M}(\sigma^\alpha) \arrow[r, "\Psi(\sigma^\alpha)"'] \arrow[ur, dotted, "\Phi'(\sigma^\alpha)"] & \mathrm{Map}^\sharp((\Delta^{J^\alpha})^\flat, T) \end{tikzcd}$$

such that for every map $d\colon \sigma^\beta \to \sigma^\alpha$, $\beta < \alpha$, the following diagram commutes

$$\begin{tikzcd} \mathcal{M}(\sigma^\alpha) \arrow[r, "\Phi'(\sigma^\alpha)"] \arrow[d, "\mathcal{M}(d)"'] & \mathrm{Map}^\sharp((\Delta^{J^\alpha})^\flat, Z) \arrow[d, "\mathrm{Res}^d"] \\ \mathcal{M}(\sigma^\beta) \arrow[r, "\Phi'(\sigma^\beta)"] & \mathrm{Map}^\sharp((\Delta^{J^\beta})^\flat, Z). \end{tikzcd}$$

By induction hypothesis, for every $e\colon \sigma^{\beta'} \to \sigma^\beta$, $\beta' < \beta < \alpha$, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{M}(\sigma^\beta) \arrow[r, "\Phi'(\sigma^\beta)"] \arrow[d, "\mathcal{M}(e)"'] & \mathrm{Map}^\sharp((\Delta^{J^\beta})^\flat, Z) \arrow[d, "\mathrm{Res}^e"] \\ \mathcal{M}(\sigma^{\beta'}) \arrow[r, "\Phi'(\sigma^{\beta'})"] & \mathrm{Map}^\sharp((\Delta^{J^{\beta'}})^\flat, Z). \end{tikzcd}$$

It follows that the composed map

$$\mathcal{M}(\sigma^\alpha) \to \prod_\beta \mathcal{M}(\sigma^\beta) \to \prod_\beta \mathrm{Map}^\sharp((\Delta^{J^\beta})^\flat, \mathrm{Map}^\flat_T(B, Z^\natural)^\natural)$$

factors through $\mathrm{Map}^\sharp((\partial\Delta^{J^\alpha})^\flat, Z)$, where in the above (finite) products, β runs over all ordinals $< \alpha$ such that there exists a (unique) injective map $\sigma^\beta \to \sigma^\alpha$. This map amalgamates with $\Phi(\sigma^\alpha)\colon \mathcal{N}(\sigma^\alpha) \to \mathrm{Map}^\sharp((\Delta^{J^\alpha})^\flat, Z)$ to give a map $\Phi'_\partial(\sigma^\alpha)\colon X^\alpha \to Z$, where

$$X^\alpha = \mathcal{N}(\sigma^\alpha)^\sharp \times (\Delta^{J^\alpha})^\flat \coprod_{\mathcal{N}(\sigma^\alpha)^\sharp \times (\partial\Delta^{J^\alpha})^\flat} \mathcal{M}(\sigma^\alpha)^\sharp \times (\partial\Delta^{J^\alpha})^\flat.$$

We have constructed the following commutative diagram

$$\begin{tikzcd} X^\alpha \arrow[r, "\Phi'_\partial(\sigma^\alpha)"] \arrow[d, hook] & Z \arrow[d, "f"] \\ \mathcal{M}(\sigma^\alpha)^\sharp \times (\Delta^{J^\alpha})^\flat \arrow[r, "\Psi(\sigma^\alpha)"'] \arrow[ur, dotted, "\Phi'(\sigma^\alpha)"] & T \end{tikzcd}$$

and we need to find the dotted arrow rendering the diagram commutative. Since the inclusion $\mathcal{N}(\sigma^\alpha) \subseteq \mathcal{M}(\sigma^\alpha)$ is anodyne, the inclusion $\mathcal{N}(\sigma^\alpha)^\sharp \subseteq \mathcal{M}(\sigma^\alpha)^\sharp$ is a trivial cofibration in $\mathcal{S}\mathrm{et}^+_\Delta$ with respect to the Cartesian model structure. The lemma then follows from the fact the trivial cofibrations in $\mathcal{S}\mathrm{et}^+_\Delta$ are stable under smash products by cofibrations [15, 3.1.4.3]. □

Lemma 2.2 has the following consequence.

Lemma 2.4. *Let K be a simplicial set, $\mathcal{C}$ be an ∞-category, $i\colon A \hookrightarrow B$ be a monomorphism of simplicial sets, $f\colon \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ be the morphism induced by i. Let $\mathcal{N}$ be an object of $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/K})^{op}}$ such that $\mathcal{N}(\sigma)$ is weakly contractible for all σ, and $\Phi\colon \mathcal{N} \to \mathrm{Map}[K, \mathrm{Fun}(B, \mathcal{C})]$ be a morphism such that $\mathrm{Map}[K, f] \circ \Phi$ factors through $a\colon \Delta^0_K \to \mathrm{Map}[K, \mathrm{Fun}(A, \mathcal{C})]$. Then there exists $b\colon \Delta^0_K \to \mathrm{Map}[K, \mathrm{Fun}(B, \mathcal{C})]$ lifting a, such that for every map $g\colon K' \to K$ and every section ν of $g^*\mathcal{N}$,*

$b \circ g$ and $g^*\Phi \circ \nu\colon K' \to \mathrm{Fun}(B, \mathcal{C})$ are homotopic over $\mathrm{Fun}(A, \mathcal{C})$. Here, in $b \circ g$, we regard b as a map $K \to \mathrm{Fun}(B, \mathcal{C})$.

Proof. By Lemma 2.1, the natural map $f^\natural\colon \mathrm{Map}^\natural(B, \mathcal{C}) \to \mathrm{Map}^\natural(A, \mathcal{C})$ is a fibration in $\mathcal{S}\mathrm{et}_\Delta^+$. Applying Lemma 2.2 (1) to the fibration $f^\natural$ and the section a, we obtain the desired b. Indeed, both $b \circ g$ and $g^*\Phi \circ \nu$ being elements of the weakly contractible Kan complex $\Gamma_{g^*\Phi, g^*a}(\mathrm{Map}[K', \mathrm{Fun}(B, \mathcal{C})])$, they are equivalent as such, which means that they are homotopic over $\mathrm{Fun}(A, \mathcal{C})$. □

3. Multisimplicial sets

In this section, we introduce several notions related to multisimplicial sets. The restricted multisimplicial nerve (Definition 3.5) of a multi-bimarked simplicial set (Definition 3.4) will play an essential role in the statements of our general theorems.

Definition 3.1 (Multisimplicial set). Let I be a set. An *I-simplicial set* is a presheaf on the category $\mathbf{\Delta}^I := \mathrm{Fun}(I, \mathbf{\Delta})$. We denote by $\mathcal{S}\mathrm{et}_\Delta^I$ the category of I-simplicial sets.

We denote by $\Delta^{n_i|i\in I}$ the I-simplicial set which, as a presheaf on $\mathbf{\Delta}^I$, is represented by $([n_i])_{i\in I}$. For an I-simplicial set S, we denote by $S_{n_i|i\in I}$ the value of S at $(n_i)_{i\in I} \in \mathbf{\Delta}^I$. A $(n_i)_{i\in I}$-*cell* of an I-simplicial set S is an element of $S_{n_i|i\in I}$. By Yoneda's lemma, there is a canonical bijection of the set $S_{n_i|i\in I}$ and the set of maps from $\Delta^{n_i|i\in I}$ to S.

Let $k \geq 0$ be an integer. A *k-simplicial set* is a $\{1, \dots, k\}$-simplicial set, namely a presheaf S on the category $\mathbf{\Delta}^k := \mathbf{\Delta}^{\{1,\dots,k\}} = \mathbf{\Delta} \times \cdots \times \mathbf{\Delta}$ (k copies).

Let $J \subseteq I$. Composition with the partial opposite functor $\mathbf{\Delta}^I \to \mathbf{\Delta}^I$ sending $(\dots, S_{j'}, \dots, S_j, \dots)$ to $(\dots, S_{j'}, \dots, S_j^{op}, \dots)$ (taking *op* for S_j when $j \in J$) defines a functor $\mathrm{op}_J^I\colon \mathcal{S}\mathrm{et}_\Delta^I \to \mathcal{S}\mathrm{et}_\Delta^I$. We define $\Delta_J^{n_i|i\in I} = \mathrm{op}_J^I \Delta^{n_i|i\in I}$. Although $\Delta_J^{n_i|i\in I}$ is isomorphic to $\Delta^{n_i|i\in I}$, it will be useful in specifying the variance of many constructions. When $I = \{1, \dots, k\}$, we use the notations op_J^k and $\Delta_J^{n_1,\dots,n_k}$.

Let $f\colon J \to I$ be a map of sets. Composition with f defines a functor $\mathbf{\Delta}^f\colon \mathbf{\Delta}^I \to \mathbf{\Delta}^J$. Composition with $\mathbf{\Delta}^f$ induces a functor $(\mathbf{\Delta}^f)^*\colon \mathcal{S}\mathrm{et}_\Delta^J \to \mathcal{S}\mathrm{et}_\Delta^I$. It has a right adjoint $(\mathbf{\Delta}^f)_*\colon \mathcal{S}\mathrm{et}_\Delta^I \to \mathcal{S}\mathrm{et}_\Delta^J$. We will now look at several special cases.

If $f\colon J \to I$ is an injective map of sets, $\mathbf{\Delta}^f$ has a right adjoint $c_f\colon \mathbf{\Delta}^J \to \mathbf{\Delta}^I$ given by $c_f(F)_i = F_j$ if $f(j) = i$ and $c_f(F)_i = [0]$ if i is not in the image of f. We have $\mathbf{\Delta}^f \circ c_f = \mathrm{id}_{\mathbf{\Delta}^J}$. Composition with c_f defines a functor $\epsilon^f\colon \mathcal{S}\mathrm{et}_\Delta^I \to \mathcal{S}\mathrm{et}_\Delta^J$, right adjoint to $(\mathbf{\Delta}^f)^*$, satisfying $\epsilon^f \circ (\mathbf{\Delta}^f)^* = \mathrm{id}_{\mathcal{S}\mathrm{et}_\Delta^J}$. If $J \subseteq I$ (resp. $J \subseteq \{1, \dots, k\}$) is a subset, we write ϵ_J^I (resp. ϵ_J^k) for the functor $\epsilon^{J\subseteq I}$ corresponding to the inclusion.

If $f\colon J \to I$ is a map with $J = \{1, \dots, k'\}$ (resp. and $I = \{1, \dots, k\}$), we will write $\epsilon^I_{f(1)\cdots f(k')}$ (resp. $\epsilon^k_{f(1)\cdots f(k')}$) for $(\mathbf{\Delta}^f)_*$ even when f is *not* injective. In particular, $(\epsilon_j^k K)_n = K_{0\cdots n\cdots 0}$, where n is at the j-th position and all other indices are 0.

Let $f\colon I \to \{1\}$. Then $\delta_I := \mathbf{\Delta}^f\colon \mathbf{\Delta} \to \mathbf{\Delta}^I$ is the diagonal map. Composition with δ_I induces a functor $\delta_I^* = (\mathbf{\Delta}^f)^*$ satisfying

$$\delta_I^* \Delta_J^{n_i|i\in I} = \left(\prod_{i\in I-J} \Delta^{n_i}\right) \times \left(\prod_{j\in J} (\Delta^{n_j})^{op}\right) =: \Delta_J^{[n_i]_{i\in I}}.$$

When $J = \emptyset$, we simply write $\Delta^{[n_i]_{i\in I}}$ for $\Delta_\emptyset^{[n_i]_{i\in I}} = \prod_{i\in I} \Delta^{n_i}$. A right adjoint $\delta_*^I\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^I$ of δ_I^* can be described as follows. A $(n_i)_{i\in I}$-cell of $\delta_*^I X$ is given by a map $\Delta^{[n_i]_{i\in I}} \to X$. For $J \subseteq I$, the twisted diagonal functor $\delta_{I,J}^* = \delta_I^* \circ \mathrm{op}_J^I\colon \mathcal{S}\mathrm{et}_\Delta^I \to \mathcal{S}\mathrm{et}_\Delta$ admits the right adjoint $\delta_*^{I,J} = \mathrm{op}_J^I \circ \delta_*^I\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^I$. For a simplicial set X, a $(n_i)_{i\in I}$-cell of $\delta_*^{I,J} X$ consists of a map $\Delta_J^{[n_i]_{i\in I}} \to X$. When $J = \emptyset$, op_J^I is the identity functor so that $\delta_{I,\emptyset}^* = \delta_I^*$, $\delta_*^{I,\emptyset} = \delta_*^I$. When $I = \{1, \dots, k\}$, we write k instead of I in the previous notations. In particular,

$$\delta_k^* = \delta_I^*\colon \mathcal{S}\mathrm{et}_\Delta^k \to \mathcal{S}\mathrm{et}_\Delta, \quad (\delta_k^* X)_n = X_{n\dots n},$$

satisfying $\delta_k^* \Delta_J^{n_1,\dots,n_k} = \Delta_J^{[n_1,\dots,n_k]}$. Moreover, according to our notations, $\delta_*^k = \epsilon^1_{1\cdots 1}$, where the index 1 is repeated k times.

We define a bifunctor

$$\boxtimes\colon \mathcal{S}\mathrm{et}_\Delta^I \times \mathcal{S}\mathrm{et}_\Delta^J \to \mathcal{S}\mathrm{et}_\Delta^{I\coprod J}$$

by the formula $S \boxtimes S' = (\boldsymbol{\Delta}^{\iota_I})^* S \times (\boldsymbol{\Delta}^{\iota_J})^* S'$, where $\iota_I\colon I \hookrightarrow I \coprod J$, $\iota_J\colon J \hookrightarrow I \coprod J$ are the inclusions. In particular, when $I = \{1, \dots, k\}$, $J = \{1, \dots, k'\}$, we have

$$\boxtimes\colon \mathcal{S}\mathrm{et}_\Delta^k \times \mathcal{S}\mathrm{et}_\Delta^{k'} \to \mathcal{S}\mathrm{et}_\Delta^{k+k'}$$

by the formula $S \boxtimes S' = (\boldsymbol{\Delta}^{\iota})^* S \times (\boldsymbol{\Delta}^{\iota'})^* S'$, where $\iota\colon \{1, \dots, k\} \hookrightarrow \{1, \dots, k + k'\}$ is the identity and $\iota'\colon \{1, \dots, k'\} \hookrightarrow \{1, \dots, k+k'\}$ sends j to $j+k$. In other words, $(S\boxtimes S')_{n_1\cdots n_{k+k'}} = S_{n_1\cdots n_k} \times S'_{n_{k+1}\cdots n_{k+k'}}$. We have $\Delta^{n_1} \boxtimes \cdots \boxtimes \Delta^{n_k} = \Delta^{n_1,\dots,n_k}$.

Definition 3.2 (Multi-marked simplicial set). An *I-marked simplicial set* (resp. *I-marked ∞-category*) is the data $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i\in I})$, where X is a simplicial set (resp. an ∞-category) and, for all $i \in I$, $\mathcal{E}_i$ is a set of edges of X which contains every degenerate edge. A morphism $f\colon (X, \{\mathcal{E}_i\}_{i\in I}) \to (X', \{\mathcal{E}'_i\}_{i\in I})$ of I-marked simplicial sets is a map $f\colon X \to X'$ having the property that $f(\mathcal{E}_i) \subseteq \mathcal{E}'_i$ for all $i \in I$. We denote the category of I-marked simplicial sets by $\mathcal{S}\mathrm{et}_\Delta^{I+}$. It is the strict fiber product of I copies of $\mathcal{S}\mathrm{et}_\Delta^{+}$ above $\mathcal{S}\mathrm{et}_\Delta$. For two I-bimarked simplicial sets $(X, \mathcal{E})$ and $(X', \mathcal{E}')$, we denote by $\mathrm{Map}^\flat((X, \mathcal{E}), (X', \mathcal{E}))$ the simplicial set such that

$$\mathrm{Map}^\flat((X, \mathcal{E}), (X', \mathcal{E}))_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta^{I+}}((\Delta^n, \mathcal{F}) \times (X, \mathcal{E}), (X', \mathcal{E}'))$$

where $\mathcal{F}_i$ consists of only degenerate edges of Δ^n for $i \in I$. When $I = \{1\}$, this agrees with the notion defined in [15, 3.1.3].

Consider the functor $\delta^*_{I+}\colon \mathcal{S}\mathrm{et}_\Delta^I \to \mathcal{S}\mathrm{et}_\Delta^{I+}$ sending S to $(\delta^*_I S, \{\mathcal{E}_i\}_{i\in I})$, where $\mathcal{E}_i$ is the set of edges of $\epsilon^I_i S \subseteq \delta^*_I S$. This functor admits a right adjoint $\delta^{I+}_*\colon \mathcal{S}\mathrm{et}_\Delta^{I+} \to \mathcal{S}\mathrm{et}_\Delta^I$ sending $(X, \{\mathcal{E}_i\}_{i\in I})$ to the I-simplicial subset of $\delta^I_* X$ whose $(n_i)_{i\in I}$-cells are maps $\Delta^{[n_i]_{i\in I}} \to X$ such that for every $i \in I$ and every map $\Delta^1 \to \epsilon^I_i \Delta^{n_i|i\in I}$, the composition

$$\Delta^1 \to \epsilon^I_i \Delta^{\{n_i\}_{i\in I}} \to \Delta^{n_i|i\in I} \to X$$

is in $\mathcal{E}_i$.

An *k-marked simplicial set* is a $\{1, \dots, k\}$-marked simplicial set. We will write k instead of I in the notations $\mathcal{S}\mathrm{et}_\Delta^{I+}$ [3], δ^*_{I+} and δ^{I+}_*.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category, we say a set $\mathcal{E}$ of edges of $\mathcal{C}$ which contains every degenerate edge is *admissible* if it satisfies the following conditions:

- $\mathcal{E}$ is *stable under pullback* [15, 6.1.3.4]: for every Cartesian diagram in the form

$$\begin{tikzcd} w \arrow[r] \arrow[d] & z \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

  in $\mathcal{C}$ with the lower horizontal arrow in $\mathcal{E}$, the upper horizontal arrow is also in $\mathcal{E}$.
- For every 2-cell $\sigma\colon \Delta^2 \to \mathcal{C}$ with $\sigma \circ d^2_0 \in \mathcal{E}$, $\sigma \circ d^2_1 \in \mathcal{E}$ if and only if $\sigma \circ d^2_2 \in \mathcal{E}$.

An I-marked ∞-category $(\mathcal{C}, \{\mathcal{E}_i\}_{i\in I})$ is said to be *admissible* if each $\mathcal{E}_i$ is admissible.

Definition 3.4 (Multi-bimarked simplicial set). An *I-bimarked simplicial sets* (resp. *I-bimarked ∞-category*) is the data $(X, \mathcal{B} = \{\mathcal{B}_{ij}\}_{i,j\in I})$, where X is a simplicial set (resp. an ∞-category) and, for all $i, j \in I$, $\mathcal{B}_{ij}$ is a subset of $\mathrm{Hom}(\Delta^1 \times \Delta^1, X)$ containing all maps factorizing through Δ^0 such that $\mathcal{B}_{ij}$ and $\mathcal{B}_{ji}$ are obtained from each other by the automorphism of $\Delta^1 \times \Delta^1$ swapping the two factors. A morphism $f\colon (X, \mathcal{B}) \to (X', \mathcal{B}')$ of I-bimarked simplicial sets is a map $f\colon X \to X'$ having the property that $f(\mathcal{B}_{ij}) \subseteq \mathcal{B}'_{ij}$ for all $i, j \in I$. We denote the category of I-bimarked simplicial sets by $\mathcal{S}\mathrm{et}_\Delta^{I++}$.

[3] In particular, $\mathcal{S}\mathrm{et}_\Delta^{2+}$ in our notation is $\mathcal{S}\mathrm{et}_\Delta^{++}$ in [16, 6.2.3.2].

If $\mathcal{E}$ and $\mathcal{E}'$ are sets of edges of X containing degenerate edges, we denote by $\mathcal{E} *_X \mathcal{E}'$ the set of maps $f\colon \Delta^1 \times \Delta^1 \to X$ such that the composition

$$\Delta^1 \simeq \Delta^1 \times \Delta^0 \xrightarrow{\mathrm{id} \times d^1_\alpha} \Delta^1 \times \Delta^1 \xrightarrow{f} X$$

is in $\mathcal{E}$ and the composition

$$\Delta^1 \simeq \Delta^0 \times \Delta^1 \xrightarrow{d^1_\alpha \times \mathrm{id}} \Delta^1 \times \Delta^1 \xrightarrow{f} X$$

is in $\mathcal{E}'$, for both $\alpha = 0, 1$. The functor $U\colon \mathcal{S}\mathrm{et}^{I+}_\Delta \to \mathcal{S}\mathrm{et}^{I++}_\Delta$ sending $(X, \{\mathcal{E}_i\}_{i\in I})$ to $(X, \{\mathcal{E}_i *_X \mathcal{E}_j\}_{i,j\in I})$ has a left adjoint $V\colon \mathcal{S}\mathrm{et}^{I++}_\Delta \to \mathcal{S}\mathrm{et}^{I+}_\Delta$ sending $(X, \mathcal{B})$ to $(X, \{\mathcal{E}_i\}_{i\in I})$, where

$$\mathcal{E}_i = \bigcup_{\substack{j\in I\\ \alpha=0,1}} \{f \circ (d^1_\alpha \times \mathrm{id}) \mid f \in \mathcal{B}_{ij}\}.$$

Consider the functor $\delta^*_{I++}\colon \mathcal{S}\mathrm{et}^I_\Delta \to \mathcal{S}\mathrm{et}^{I++}_\Delta$ sending S to $(\delta^*_I S, \mathcal{B})$, where $\mathcal{B}_{ij} = (\epsilon^I_{ij} S)_{11}$. The functor δ^*_{I++} has a right adjoint $\delta^{I++}_*\colon \mathcal{S}\mathrm{et}^{I++}_\Delta \to \mathcal{S}\mathrm{et}^I_\Delta$ sending $(X, \mathcal{B})$ to the I-simplicial subset of $\delta^I_* X$ whose $(n_i)_{i\in I}$-cells are maps $\Delta^{[n_i]_{i\in I}} \to X$ such that for all $i, j \in I$, and every $(1,1)$-cell of $\epsilon^I_{ij}\Delta^{n_i|i\in I}$, the composition

$$\Delta^1 \times \Delta^1 \to \epsilon^I_{ij}\Delta^{n_i|i\in I} \to \Delta^{[n_i]_{i\in I}} \to X$$

is in $\mathcal{B}_{ij}$. We have $\delta^*_{I+} = V \circ \delta^*_{I++}$ and $\delta^{I+}_* = \delta^{I++}_* \circ U$.

Definition 3.5 (Restricted multisimplicial nerve). Let $(X, \mathcal{B})$ be an I-bimarked simplicial set, then the I-simplicial set $\delta^{I++}_*(X, \mathcal{B})$ is called the *restricted I-simplicial nerve* of X.

An *k-bimarked simplicial set* is a $\{1, \dots, k\}$-bimarked simplicial set. We will then write k instead of I in the corresponding notations.

Definition 3.6. Let $\mathcal{C}$ an ∞-category, we say a subset $\mathcal{B}$ of $\mathrm{Hom}(\Delta^1 \times \Delta^1, \mathcal{C})$ (containing all maps factorizing through Δ^0) is *vertically admissible* if it satisfies the following conditions:

- $\mathcal{B}$ is *stable under pullback*: for every cube $\Delta^1 \times \Delta^1 \times \Delta^1 \to \mathcal{C}$ as in

$$\begin{tikzcd}
& w \arrow[rr] \arrow[dd] & & z \arrow[dd] \\
w' \arrow[ur] \arrow[rr] \arrow[dd] & & z' \arrow[ur] \arrow[dd] & \\
& y \arrow[rr] & & x \\
y' \arrow[ur] \arrow[rr] & & x' \arrow[ur] &
\end{tikzcd}$$

  with the back square in $\mathcal{B}$ and all other squares except the front one being Cartesian diagrams, the front square is in $\mathcal{B}$.
- $\mathcal{B}$ is *stable under horizontal composition*: for every map $\Delta^2 \times \Delta^1 \to \mathcal{C}$ as

$$\begin{tikzcd}
z' \arrow[r] \arrow[d] & y' \arrow[r] \arrow[d] & x' \arrow[d] \\
z \arrow[r] & y \arrow[r] & x
\end{tikzcd}$$

  with both inner squares in $\mathcal{B}$, the outer square is also in $\mathcal{B}$.

- For every map $\Delta^2 \times \Delta^1 \to \mathcal{C}$ as
$$\begin{tikzcd} y'' \arrow[r] \arrow[d] & x'' \arrow[d] \\ y' \arrow[r] \arrow[d] & x' \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$
with the lower square in $\mathcal{B}$, the upper square is in $\mathcal{B}$ if and only if the outer square is in $\mathcal{B}$.

We say $\mathcal{B}$ is *horizontally admissible* if the set $\mathcal{B}'$ obtained by transposing the squares in $\mathcal{B}$ is vertically admissible. We say $\mathcal{B}$ is *biadmissible* if it is both vertically admissible and horizontally admissible.

Notation 3.7. Let S be a simplicial set and $(\mathcal{C},\mathcal{B})$ be an I-bimarked ∞-category, we denote by $\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}))$ the I-bimarked ∞-category $(\mathrm{Map}^\flat(S^\sharp,\mathcal{C}^\natural),\mathrm{Map}^\natural(S,\mathcal{B}))$ where $\mathrm{Map}^\natural(S,\mathcal{B})_{ij}$ consists of maps $F\colon S\times\Delta^1\times\Delta^1\to\mathcal{C}$ such that $F\mid\{s\}\times\Delta^1\times\Delta^1$ is in $\mathcal{B}_{ij}$ for every vertex s of S. In particular, if $(\mathcal{C},\mathcal{B})$ is biadmissible (resp. vertically admissible, resp. horizontally admissible), then $\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}))$ is biadmissible (resp. vertically admissible, resp. horizontally admissible).

Notation 3.8.

(1) For an I-marked simplicial set $(X,\mathcal{E}=\{\mathcal{E}_i\}_{i\in I})$, we simply let
$$X_{\mathcal{E}}=X_{\{\mathcal{E}_i\}_{i\in I}}=\delta^{I+}_*(X,\{\mathcal{E}_i\}_{i\in I}).$$
In particular, for any marked simplicial set $(X,\mathcal{E})$, $X_{\mathcal{E}}\subseteq X$ is the simplicial subset spanned by the edges in $\mathcal{E}$.

(2) If $\mathcal{C}$ is an ∞-category and $\mathcal{E}$ and $\mathcal{E}'$ are sets of edges of $\mathcal{C}$ containing degenerate edges, we denote by $\mathcal{E}*^{\mathrm{cart}}_{\mathcal{C}}\mathcal{E}'$ the subset of $\mathcal{E}*_{\mathcal{C}}\mathcal{E}'$ consisting of maps $f\colon\Delta^1\times\Delta^1\to\mathcal{C}$ which are Cartesian diagrams.

(3) For an I-marked ∞-category $(\mathcal{C},\mathcal{E})$, we denote by $(\mathcal{C},\mathcal{E}^{\mathrm{cart}})$ the I-bimarked ∞-category such that $\mathcal{E}^{\mathrm{cart}}_{ii}=\mathcal{E}_i*_{\mathcal{C}}\mathcal{E}_i$ and $\mathcal{E}^{\mathrm{cart}}_{ii'}=\mathcal{E}_i*^{\mathrm{cart}}_{\mathcal{C}}\mathcal{E}_{i'}$ for $i,i'\in I$ and $i\neq i'$.

(4) We let
$$\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}}=\delta^{I++}_*(\mathcal{C},\mathcal{E}^{\mathrm{cart}})$$
be the *Cartesian I-simplicial nerve* of $(\mathcal{C},\mathcal{E})$. When $\mathcal{E}_i=\mathcal{C}_1$ for every $i\in I$, we simply write $\mathcal{C}^{\mathrm{cart}}_{|I}$ for $\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}}$.

Remark 3.9. Let $\mathcal{C}$ be a 2-category, namely, a category enriched in the category of categories. We regard $\mathcal{C}$ as a simplicial category by taking $\mathrm{N}(\mathrm{Map}_{\mathcal{C}}(X,Y))$ for all objects X and Y of $\mathcal{C}$. Assume that $\mathcal{C}$ is a $(2,1)$-category, namely, that all 2-cells of $\mathcal{C}$ are invertible. Then the *simplicial nerve* $\mathrm{N}(\mathcal{C})$ [15, 1.1.5.5] is an ∞-category. Let $\mathcal{E}_1,\dots,\mathcal{E}_k$ be sets of morphisms of $\mathcal{C}$ stable under composition and containing identity morphisms. Then $\mathrm{N}(\mathcal{C})_{\mathcal{E}_1,\dots,\mathcal{E}_k}$ and $\mathrm{N}(\mathcal{C})^{\mathrm{cart}}_{\mathcal{E}_1,\dots,\mathcal{E}_k}$ can be interpreted as the k-fold nerves in the sense of Fiore and Paoli [4, 2.14] of suitable k-fold categories.

4. Multisimplicial descent

In this section, we study the map of obtained by composing maps in two directions in a multisimplicial nerve. Unlike in Theorem 0.1, the two directions are not subject to the Cartesian restriction. The main result is Theorem 4.4, which can be regarded a generalization of Deligne's theorem [3, 3.3.2] (see Corollary 4.7).

Recall that a *partially ordered set* P is an (ordinary) category such that there is at most one arrow (usual denoted as $\leq$) between each pair of objects. For every element $p\in P$, we identify the overcategory $P_{/p}$ (resp. undercategory $P_{p/}$) with the full partially ordered subset of P consisting of elements $\leq p$ (resp. $\geq p$). In particular, for $p,p'\in P$, $P_{p//p'}$ is identified with the full partially ordered subset of P consisting of elements both $\geq p$ and $\leq p'$, which is empty unless $p\leq p'$.

We enumerate the vertices of the bisimplicial set $\Delta^{n,n}$ by coordinates (i,j) for $0\leq i,j\leq n$. The isomorphism $\delta_2^*\Delta^{n,n}\simeq\Delta^n\times\Delta^n\simeq\mathrm{N}([n]\times[n])$ induces a corresponding enumeration on the partially

ordered set $[n]\times[n]$, such that $(i,j)\leq(i',j')$ if and only if $i\leq i'$ and $j\leq j'$. We define, for every $n\geq 0$, $\mathbf{Cpt}^n\subseteq\Delta^{n,n}$ as the bisimplicial subset generated by the vertices (i,j) with $0\leq i\leq j\leq n$. Identify $\delta_2^*\mathbf{Cpt}^n$ with a simplicial subset of $\mathrm{N}([n]\times[n])$ through the embedding $\delta_2^*\mathbf{Cpt}^n\hookrightarrow\delta_2^*\Delta^{n,n}\simeq\mathrm{N}([n]\times[n])$. It is easy to see that

$$\delta_2^*\mathbf{Cpt}^n=\square^n\subseteq\mathcal{C}\mathrm{pt}^n:=\mathrm{N}(\mathrm{Cpt}^n)\subseteq\mathrm{N}([n]\times[n]),$$

where

- Cpt^n is the partially ordered subset of $[n]\times[n]$ generated by (i,j) with $0\leq i\leq j\leq n$;
- $\square^n=\bigcup_{k=0}^n\square^n_k$, where $\square^n_k$ is the nerve of the partially ordered subset of $[n]\times[n]$ generated by (i,j) with $0\leq i\leq k\leq j\leq n$.

The following lemma is crucial for our argument. Its proof will be given in Lemma 7.4.

Lemma 4.1. *The inclusion $\square^n\subseteq\mathcal{C}\mathrm{pt}^n$ is inner anodyne.*

Let K be a set, $(X,\mathcal{B})$ be a $(\{1,2\}\coprod K)$-bimarked simplicial set. For $n,n_k\geq 0$ $(k\in K)$, σ a $(n,n_k)_{k\in K}$-cell of $\delta_*^{\{0\}\coprod K}X$, and $\alpha\in\{1,2\}\coprod K$, we define $\mathcal{K}\mathrm{omp}^\alpha(\sigma)=\mathcal{K}\mathrm{omp}^\alpha_{X,\mathcal{B}}(\sigma)$, the α-th simplicial set of compactifications of σ, as the limit of the diagram

$$\begin{tikzcd}
& & \epsilon_\alpha^{\{1,2\}\coprod K}\mathrm{Map}(\mathbf{Cpt}^n\boxtimes\Delta^{n_k|_{k\in K}},\delta_*^{(\{1,2\}\coprod K)++}(X,\mathcal{B})) \arrow[d, hook, "g"] \\
& \mathrm{Map}(\mathcal{C}\mathrm{pt}^n\times\Delta^{[n_k]_{k\in K}},X) \arrow[r, "\mathrm{res}_1"] \arrow[d, "\mathrm{res}_2"] & \mathrm{Map}(\square^n\times\Delta^{[n_k]_{k\in K}},X) \\
\{\sigma\} \arrow[r, hook] & \mathrm{Map}(\Delta^{[n,n_k]_{k\in K}},X) &
\end{tikzcd}$$

in the category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets, where res_1 is induced by the inclusion $\square^n\subseteq\mathcal{C}\mathrm{pt}^n$, res_2 is induced by the diagonal map $\Delta^n\to\mathcal{C}\mathrm{pt}^n$, g is the composition of the commutative diagram

$$\begin{tikzcd}
\epsilon_\alpha^{\{1,2\}\coprod K}\mathrm{Map}(\mathbf{Cpt}^n\boxtimes\Delta^{n_k|_{k\in K}},\delta_*^{(\{1,2\}\cup K)++}(X,\mathcal{B})) \arrow[r, hook] \arrow[d, "\delta^*_{\{1,2\}\coprod K}"'] & \epsilon_\alpha^{\{1,2\}\coprod K}\mathrm{Map}(\mathbf{Cpt}^n\boxtimes\Delta^{n_k|_{k\in K}},\delta_*^{\{1,2\}\coprod K}X) \arrow[d, "\simeq"] \\
\mathrm{Map}(\square^n\times\Delta^{[n_k]_{k\in K}},\delta^*_{\{1,2\}\coprod K}\delta_*^{(\{1,2\}\coprod K)++}(X,\mathcal{B})) \arrow[r] & \mathrm{Map}(\square^n\times\Delta^{[n_k]_{k\in K}},X).
\end{tikzcd}$$

We denote by $\phi(\sigma)$ the map $\mathcal{K}\mathrm{omp}^\alpha(\sigma)\to\mathrm{Map}(\square^n\times\Delta^{[n_k]_{k\in K}},\delta^*_{\{1,2\}\coprod K}\delta_*^{(\{1,2\}\coprod K)++}(X,\mathcal{B}))$ and by $\psi(\sigma)$ the projection $\mathcal{K}\mathrm{omp}^\alpha(\sigma)\to\mathrm{Map}(\mathcal{C}\mathrm{pt}^n\times\Delta^{[n_k]_{k\in K}},X)$. If $\mathcal{E}$ is a set of edges of X which factorize through an element of $\mathcal{B}_{ij}$, $i,j\in\{1,2\}\coprod K$, then $(X',\mathcal{B})$ is a $(\{1,2\}\coprod K)$-bimarked simplicial set, where $X'=X_{\mathcal{E}}\subseteq X$. If, moreover, $(X,\mathcal{E})\to(\Delta^0)^\flat$ has the right lifting property with respect to $(\Lambda^2_1)^\sharp\coprod_{(\Lambda^2_1)^\flat}(\Delta^2)^\flat\subseteq(\Delta^2)^\sharp$, then $\mathcal{K}\mathrm{omp}^\alpha_{X',\mathcal{B}}(\sigma)\simeq\mathcal{K}\mathrm{omp}^\alpha_{X,\mathcal{B}}(\sigma)$ for every cell σ of $\delta_*^{\{0\}\coprod K}X'$.

Remark 4.2.

(1) Let $(X,\mathcal{E}_1,\mathcal{E}_2,\{\mathcal{E}_k\}_{k\in K})$ be a $(\{1,2\}\coprod K)$-marked simplicial set, σ be a $(n,n_k)_{k\in K}$-cell of $\delta_*^{\{0\}\coprod K}X$. We put

$$\mathcal{K}\mathrm{omp}^\alpha_{X,\mathcal{E}_1,\mathcal{E}_2,\{\mathcal{E}_k\}_{k\in K}}(\sigma)=\mathcal{K}\mathrm{omp}^\alpha_{X,\mathcal{B}}(\sigma),$$

where $(X,\mathcal{B})=U(X,\mathcal{E}_1,\mathcal{E}_2,\{\mathcal{E}_k\}_{k\in K})$. If $\sigma\notin(\delta_*^{(\{0\}\coprod K)+}(X,X_1,\{\mathcal{E}_k\}_{k\in K}))_{n,n_k|k\in K}$, then $\mathcal{K}\mathrm{omp}^\alpha(\sigma)=\emptyset$ for all $\alpha\in\{1,2\}\coprod K$. Assume that $\sigma\in(\delta_*^{(\{0\}\coprod K)+}(X,X_1,\{\mathcal{E}_k\}_{k\in K}))_{n,n_k|k\in K}$. Let $Y=\mathrm{Map}^\flat(\delta^*_{K+}\Delta^{n_k|k\in K},(X,\{\mathcal{E}_k\}_{k\in K}))$,

$$\mathcal{F}_\alpha=\mathrm{Hom}(\delta^*_{(\{\alpha\}\coprod K)+}\Delta^{1,n_k|k\in K},(X,\mathcal{E}_\alpha,\{\mathcal{E}_k\}_{k\in K})),\quad\alpha=1,2.$$

Then σ induces an n-cell τ of Y and $\mathcal{K}\mathrm{omp}^{\alpha}_{X,\mathcal{E}_1,\mathcal{E}_2,\{\mathcal{E}_k\}_{k\in K}}(\sigma)\simeq\mathcal{K}\mathrm{omp}^{\alpha}_{Y,\mathcal{F}_1,\mathcal{F}_2}(\tau)$, $\alpha=1,2$.

(2) Let $(\mathcal{C},\mathcal{E}_1,\mathcal{E}_2)$ be a 2-marked ∞-category, $\alpha=1,2$. Assume that $(\mathcal{C},\mathcal{E}_\alpha)\to(\Delta^0)^\flat$ has the right lifting property with respect to

$$(\Lambda^2_1)^\sharp\coprod_{(\Lambda^2_1)^\flat}(\Delta^2)^\flat\subseteq(\Delta^2)^\sharp.$$

Then $\mathcal{K}\mathrm{omp}^{\alpha}(\sigma)$ is an ∞-category for every simplex σ of $\mathcal{C}$. Since we do not use this fact in the sequel, we omit the proof.

Lemma 4.3. *Let $f\colon Y\to Z$ be a map of simplicial sets. Assume that for every $l=0,1,2$, every ∞-category $\mathcal{D}$ and every commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "f"'] & \mathrm{Fun}(\Delta^l,\mathcal{D}) \arrow[d, "p"] \\ Z \arrow[r, "w"] & \mathrm{Fun}(\partial\Delta^l,\mathcal{D}) \end{tikzcd}$$

where p is induced by the inclusion $\partial\Delta^l\subseteq\Delta^l$, there exists a map $u\colon Z\to\mathrm{Fun}(\Delta^l,\mathcal{D})$ such that $p\circ u=w$ and that $u\circ f$ and v are homotopic over $\mathrm{Fun}(\partial\Delta^l,\mathcal{D})$. Then f is a categorical equivalence.

Proof. Let $\mathcal{D}$ be an ∞-category. The assumption for $l=0,1,2$ implies that the functor

$$\mathrm{hFun}(Z,\mathcal{D})\to\mathrm{hFun}(Y,\mathcal{D})$$

induced by f is essentially surjective, full, and faithful, respectively. Thus it is an equivalence of categories. It follows from [15, 2.2.5.8] that f is a categorical equivalence. $\square$

Theorem 4.4 (Multisimplicial descent)**.** *Let K be a set, $(X,\mathcal{B})$ be a $(\{1,2\}\coprod K)$-bimarked simplicial set, $(X,\mathcal{B}')$ be a $(\{0\}\coprod K)$-bimarked simplicial set such that $\delta_*^{(\{1,2\}\coprod K)++}(X,\mathcal{B})\subseteq(\boldsymbol{\Delta}^\mu)_*\delta_*^{(\{0\}\coprod K)++}(X,\mathcal{B}')$, where $\mu\colon\{1,2\}\coprod K\to\{0\}\coprod K$ sends $1,2$ to 0 and k to k for $k\in K$. Let $\alpha\in\{1,2\}\coprod K$, $L\subseteq K$. Assume that $\mathcal{K}\mathrm{omp}^{\alpha}_{X,\mathcal{B}}(\sigma)$ is weakly contractible for all $n\geq0$ and all $\sigma\in(\delta_*^{(\{0\}\coprod K)++}(X,\mathcal{B}'))_{n,n_k|k\in K}$ with $n_k=n$. Then the map*

$$\begin{aligned} f\colon \delta^*_{\{1,2\}\coprod K,L}\delta_*^{(\{1,2\}\coprod K)++}(X,\mathcal{B}) &\simeq \delta^*_{\{0\}\coprod K,L}(\boldsymbol{\Delta}^\mu)^*\delta_*^{(\{1,2\}\cup K)++}(X,\mathcal{B})\\ &\to \delta^*_{\{0\}\coprod K,L}\delta_*^{(\{0\}\coprod K)++}(X,\mathcal{B}') \end{aligned}$$

is a categorical equivalence.

Proof. Let Y and Z be the source and target of f, respectively. Consider a commutative diagram as in Lemma 4.3. For every n-cell σ of Z, corresponding to an $(n,n_k)_{k\in K}$-cell τ of $\delta_*^{(\{0\}\coprod K)++}(X,\mathcal{B}')$, where $n_k=n$, consider the commutative diagram

$$\begin{tikzcd} \mathcal{N}(\sigma) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^l\times\mathcal{C}\mathrm{pt}^n\times\Delta_L^{[n_k]_{k\in K}},\mathcal{D}) \arrow[r, "\mathrm{res}_2"] \arrow[d, "\mathrm{res}_1"] & \mathrm{Fun}(\Delta^l\times\Delta^n,\mathcal{D}) \arrow[d] \\ \mathcal{K}\mathrm{omp}^{\alpha}(\tau) \arrow[r, "h"] & \mathrm{Fun}(H\times\Delta_L^{[n_k]_{k\in K}},\mathcal{D}) \arrow[r, "\mathrm{res}_2"] & \mathrm{Fun}(\partial\Delta^l\times\Delta^n,\mathcal{D}) \end{tikzcd}$$

where res_1 is induce by

$$i\colon H=\Delta^l\times\square^n\coprod_{\partial\Delta^l\times\square^n}\partial\Delta^l\times\mathcal{C}\mathrm{pt}^n\hookrightarrow\Delta^l\times\mathcal{C}\mathrm{pt}^n,$$

h is the amalgamation of $v_*\phi(\tau)$ and $w_*\psi(\tau)$, the square on the left is Cartesian, and the maps res_2 are induced by the diagonal embedding $\Delta^n\subseteq\Delta_L^{[n_k]_{k\in K}}\times\mathcal{C}\mathrm{pt}^n$. By Lemma 4.1 and [15, 2.3.2.4, 2.3.2.5], $i\times\mathrm{id}$ is inner anodyne and consequently res_1 is a trivial fibration. Thus $\mathcal{N}(\sigma)$ is weakly contractible.

The composition of the lower horizontal arrows is constant of value $w(\sigma)$. Let us denote by $\Phi(\sigma)$ the composition of the upper horizontal arrows. Since f induces an isomorphism on vertices, the image of $\Phi(\sigma)$, when restricted to $(\Delta^n)_0 \times \Delta^l$, is constant of value $v(\sigma_0)$. In particular, the image of σ is contained in $\mathrm{Map}^\sharp((\Delta^n)^\flat, \mathrm{Fun}(\Delta^l, \mathcal{D})^\natural)$. This construction is functorial in σ, giving rise to a morphism $\Phi\colon \mathcal{N} \to \mathrm{Map}[Z, \mathrm{Fun}(\Delta^l, \mathcal{D})]$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/Z})^{op}}$. Moreover, for every n-cell σ' of Y corresponding to a $(n, n, n_k)_{k\in K}$-cell of $\delta_*^{(\{1,2\}\coprod K)++}(X, \mathcal{B})$, there is a canonical vertex $\nu(\sigma')$ of $\mathcal{N}(f(\sigma'))$, functorial in σ', whose image under $\Phi(f(\sigma'))$ is $v(\sigma')$. Applying Lemma 2.2 to Φ and ν, we obtain $u\colon Z \to \mathrm{Fun}(\Delta^l, \mathcal{D})$ such that $u \circ f$ and v are homotopic over $\mathrm{Fun}(\partial\Delta^l, \mathcal{D})$, as desired. □

Corollary 4.5. *Let $\mathcal{C}$ be an ∞-category admitting pullbacks, K be a* finite *set, and $(\mathcal{C}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \{\mathcal{E}_k\}_{k\in K})$ be a $(\{0,1,2\}\coprod K)$-marked ∞-category such that $\mathcal{E}_k$ $(k = 0, 1, 2$ or $k \in K)$ is stable under composition and pullback, $\mathcal{E}_1 *_{\mathcal{C}} \mathcal{E}_2 \subseteq \mathrm{Hom}((\Delta^1 \times \Delta^1)^\sharp, (\mathcal{C}, \mathcal{E}_0))$. Let $\alpha \in \{1, 2\}$, $L \subseteq K$. Assume that for every simplex σ of $\mathcal{C}_{\mathcal{E}_0} \subseteq \mathcal{C}$, $\mathcal{K}\mathrm{omp}^\alpha_{\mathcal{C},\mathcal{E}_1,\mathcal{E}_2}(\sigma)$ is weakly contractible. Then the map*

$$f\colon \delta^*_{\{1,2\}\coprod K, L}\delta_*^{(\{1,2\}\coprod K)++}(\mathcal{C}, \mathcal{B}) \to \delta^*_{\{0\}\coprod K, L}\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_0, \{\mathcal{E}_k\}_{k\in K}}$$

is a categorical equivalence, where $\mathcal{B}$ is determined by $\mathcal{B}_{ij} = (\mathcal{E}_1, \mathcal{E}_2, \{\mathcal{E}_k\}_{k\in K})^{\mathrm{cart}}_{ij}$ if $(i, j) \neq (1, 2), (2, 1)$ and $\mathcal{B}_{12} = \mathcal{E}_1 *_{\mathcal{C}} \mathcal{E}_2$.

Proof. By Theorem 4.4, it suffices to show that for every $(n, n_k)_{k\in K}$-cell $(n_k = n)$ σ of the $(\{0\}\coprod K)$-simplicial set $\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_0,\{\mathcal{E}_k\}_{k\in K}}$, $\mathcal{K}\mathrm{omp}^\alpha_{\mathcal{C},\mathcal{B}}(\sigma)$ is weakly contractible. We have the following Cartesian square

$$\begin{tikzcd}
\mathcal{K}\mathrm{omp}^\alpha_{\mathcal{C},\mathcal{B}}(\sigma) \arrow[rr] \arrow[d] & & \mathcal{K}\mathrm{omp}^\alpha_{\mathcal{C},\mathcal{E}_1,\mathcal{E}_2}(\tau) \arrow[d] \\
\mathrm{Fun}(\mathcal{C}\mathrm{pt}^n \times \Delta^{[n_k]_{k\in K}}, \mathcal{C})_{\mathrm{RKE}} \arrow[r, "\mathrm{res}_1"] & \mathrm{Fun}(\mathrm{N}(Q\coprod_P R), \mathcal{C}) \arrow[r, "\mathrm{res}_2"] & \mathrm{Fun}(\mathrm{N}(Q)\coprod_{\mathrm{N}(P)}\mathrm{N}(R), \mathcal{C}),
\end{tikzcd}$$

where

- $P = [n] \times \{\infty\}$ where ∞ is the final object of $[n]^K$;
- $Q = [n] \times [n]^K \subseteq \mathrm{Cpt}^n \times [n]^K$ is the diagonal inclusion;
- $R = \mathrm{Cpt}^n \times \{\infty\}$;
- τ is the n-cell of $\mathcal{C}$, restriction of σ to $\Delta^n \times \{\infty\}$;
- $\mathrm{Fun}(\mathcal{C}\mathrm{pt}^n \times \Delta^{[n_k]_{k\in K}}, \mathcal{C})_{\mathrm{RKE}} \subseteq \mathrm{Fun}(\mathcal{C}\mathrm{pt}^n \times \Delta^{[n_k]_{k\in K}}, \mathcal{C})$ is the full subcategory spanned by functors $F\colon \mathcal{C}\mathrm{pt}^n \times \Delta^{[n_k]_{k\in K}} \to \mathcal{C}$ which are right Kan extensions of $F \mid \mathrm{N}(Q\coprod_P R)$.

Since $Q\coprod_P R \subseteq \mathrm{Cpt}^n \times [n]^K$ is a full partially ordered subset, res_1 is a trivial fibration by [15, 4.3.2.15]. Since $\mathcal{C}$ admits pullbacks, res_2 is a trivial fibration by Lemma 7.2. It follows that the upper horizontal arrow is a trivial fibration. □

Let $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$ be a 2-marked (ordinary) category, $1 \leq \alpha \leq 2$. Assume that $\mathcal{E}_\alpha$ is stable under composition. For $n \geq 0$ and an n-cell $\mathbf{c}_n = (c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n)$ in $\mathrm{N}(\mathcal{C})$, $\mathcal{K}\mathrm{omp}^\alpha(\mathbf{c}_n) = \mathcal{K}\mathrm{omp}^\alpha_{\mathrm{N}(\mathcal{C}),\mathcal{E}_1,\mathcal{E}_2}(\mathbf{c}_n) \simeq \mathrm{N}(\mathrm{Comp}^\alpha(\mathbf{c}_n))$, where $\mathrm{Comp}^\alpha(\mathbf{c}_n)$ is the category whose objects are maps

$\gamma\colon \mathbf{Cpt}^n \to \mathrm{N}(\mathcal{C})_{\mathcal{E}_1,\mathcal{E}_2}$ such that $p_i \circ q_i = f_i$ for $i = 1, \ldots, n$ as in the following diagram

$$\begin{tikzcd}
c_0 \arrow[r, "q_1"] & \bullet \arrow[r] \arrow[d, "p_1"'] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& c_1 \arrow[r, "q_2"] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
& & \ddots & \vdots \arrow[d, "p_{n-1}"'] & \vdots \arrow[d] \\
& & & c_{n-1} \arrow[r, "q_n"] & \bullet \arrow[d, "p_n"'] \\
& & & & c_n .
\end{tikzcd}$$

When $\alpha = 1$ (resp. $\alpha = 2$), morphisms from γ_0 to γ_1 are maps $\tilde{\gamma}\colon \mathbf{Cpt}^n \times \Delta^{1,0} \to \mathrm{N}_{\mathcal{E}_1,\mathcal{E}_2}(\mathcal{C})$ (resp. $\tilde{\gamma}\colon \mathbf{Cpt}^n \times \Delta^{0,1} \to \mathrm{N}_{\mathcal{E}_1,\mathcal{E}_2}(\mathcal{C})$) such that $\tilde{\gamma} \mid \mathbf{Cpt}^n \times \{(0,0)\} = \gamma_0$, $\tilde{\gamma} \mid \mathbf{Cpt}^n \times \{(1,0)\} = \gamma_1$ (resp. $\tilde{\gamma} \mid \mathbf{Cpt}^n \times \{(0,1)\} = \gamma_1$), and $\tilde{\gamma} \mid \{(i,j)\} \times \Delta^{1,0}$ (resp. $\tilde{\gamma} \mid \{(i,j)\} \times \Delta^{0,1}$) is of the form

$$\begin{tikzcd}
\bullet \arrow[d, "p"'] & \text{(resp. } \bullet \arrow[r, "q"] & \bullet \text{)} \\
\bullet & &
\end{tikzcd}$$

where $p = \mathrm{id}$ (resp. $q = \mathrm{id}$) if $i = j$. Composition of morphisms is defined in the obvious way.

Lemma 4.6. *Let $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$ be a 2-marked category. Assume the following three conditions:*

(1) *Every morphism f of $\mathcal{C}$ is of the form $p \circ q$, where p is a morphism of $\mathcal{E}_1$ and q is a morphism of $\mathcal{E}_2$.*
(2) *$\mathcal{E}_1$ and $\mathcal{E}_2$ are stable under composition.*
(3) *Fiber products exist in the category $(\mathrm{Ob}(\mathcal{C}), \mathcal{E}_1)$ and are fiber products in $\mathcal{C}$.*

Then the category $\mathrm{Comp}^1(\mathbf{c}_n)^{op}$ is filtered. In particular, $\mathcal{K}\mathrm{omp}^1(\mathbf{c}_n)$ is weakly contractible.

Assumption (3) is satisfied, for example, when $\mathcal{C}$ admits pullbacks and $\mathcal{E}_1$ is admissible (Definition 3.3).

Proof. We proceed by induction on n. The case $n = 0$ is trivial.

For $n \geq 1$, let us first show that $\mathrm{Comp}^1(\mathbf{c}_n)$ is nonempty. By induction hypothesis, there exists an object γ of $\mathrm{Comp}^1(\mathbf{c}_{n-1})$. Here $\mathbf{c}_{n-1} = \mathbf{c}_n \circ d^n_n$. We regard $\mathbf{Cpt}^{n-1} \subseteq \mathbf{Cpt}^n$ as the bisimplicial subset generated by (i,j), $0 \leq i \leq j \leq n-1$. For every $0 \leq k \leq n$, we define $\mathbf{Cpt}^{n-1}_k \subseteq \mathbf{Cpt}^n$ to be the bisimplicial subset generated by (i,j), $0 \leq i \leq j \leq n-1$ and (i', n), $k \leq i' \leq n$. Therefore, $\mathbf{Cpt}^{n-1} \subseteq \mathbf{Cpt}^{n-1}_n \subseteq \cdots \subseteq \mathbf{Cpt}^{n-1}_0 = \mathbf{Cpt}^n$. We extend γ to $\mathbf{Cpt}^n_k$ by descending induction on k. The case $k = n$ is trivial. Assume that γ has been extended to $\mathbf{Cpt}^{n-1}_k$ for $0 < k \leq n$. By assumption (1), $\gamma((k-1, n-1) \to (k,n))$ decomposes into

$$\begin{tikzcd}
\gamma((k-1,n-1)) \arrow[r, "q"] & X \arrow[d, "p"] \\
& \gamma((k,n)),
\end{tikzcd}$$

where p is in $\mathcal{E}_1$ and q is in $\mathcal{E}_2$. By (2), these morphisms extend γ to $\mathbf{Cpt}^{n-1}_{k-1}$ with $\gamma((k-1,n)) = X$.

Next consider two objects γ, γ' in $\mathrm{Comp}^1(\mathbf{c}_n)$ for some $n \geq 1$. We want to find another object γ'' and morphisms $\tilde{\gamma}\colon \gamma'' \to \gamma$, $\tilde{\gamma}'\colon \gamma'' \to \gamma'$ connecting them. By induction hypothesis, there is an object γ''_- of $\mathrm{Comp}^1(\mathbf{c}_{n-1})$, and morphisms $\tilde{\gamma}_- : \gamma''_- \to \gamma \mid \mathbf{Cpt}^{n-1}$, $\tilde{\gamma}'_- : \gamma''_- \to \gamma' \mid \mathbf{Cpt}^{n-1}$. We extend γ''_- and

morphisms $\tilde{\gamma}_-$, $\tilde{\gamma}'_-$ to $\mathbf{Cpt}^{n-1}_k$ by induction on k. The case $k=n$ is trivial. Assume that they have already been extended to $\mathbf{Cpt}^{n-1}_k$ for some $0<k\le n$. Consider the natural morphism

$$f\colon \gamma''_-((k-1,n-1))\to\gamma((k-1,n))\underset{\gamma((k,n))}{\times}\gamma''_-((k,n))\underset{\gamma'((k,n))}{\times}\gamma'((k-1,n)) \tag{4.1}$$

in the category $\mathcal{C}$. Choose a decomposition $f=p\circ q$ and let the middle object be $\gamma''_-((k-1,n))$. By assumption (2), the three projections in the fiber product in (4.1) are all in $\mathcal{E}_1$. Composing p with these three projections will extend γ''_-, $\tilde{\gamma}_-$ and $\tilde{\gamma}'_-$ to $\mathbf{Cpt}^{n-1}_{k-1}$.

Now consider two morphisms $\tilde{\gamma}^1,\tilde{\gamma}^2\colon\gamma'\to\gamma$ in $\mathrm{Comp}^1(\mathbf{c}_n)$. We want to find another element γ'' and a morphism $\tilde{\gamma}^0\colon\gamma''\to\gamma'$ such that $\tilde{\gamma}^1\circ\tilde{\gamma}^0=\tilde{\gamma}^2\circ\tilde{\gamma}^0$. By induction, we already have such $\tilde{\gamma}^0_-\colon\gamma''_-\to\gamma'\mid\mathbf{Cpt}^{n-1}$ when restricted on $\mathbf{Cpt}^{n-1}$. As before, We extend γ''_- and $\tilde{\gamma}^0_-$ to $\mathbf{Cpt}^{n-1}_k$ by induction on k. The case $k=n$ is trivial. Assume that they have already been extended to $\mathbf{Cpt}^{n-1}_k$ for some $0<k\le n$. By assumption (2), equalizers exist in $(\mathrm{Ob}(\mathcal{C}),\mathcal{E}_1)$ and are equalizers in $\mathcal{C}$, and we denote $\mathrm{eq}(k-1,n)$ to be the equalizer of $(\tilde{\gamma}^1,\tilde{\gamma}^2)\mid(k-1,n)$. Then there is a natural morphism

$$f\colon\gamma''_-((k-1,n-1))\to\mathrm{eq}(k-1,n)\underset{\gamma'((k,n))}{\times}\gamma''_-((k,n)) \tag{4.2}$$

in the category $\mathcal{C}$. Choose a decomposition $f=p\circ q$ and let the middle object be $\gamma''_-((k-1,n))$. It is easy to check that the two projections in the fiber product in (4.2) are both in $\mathcal{E}_1$. Composing p with these two projections will extend γ''_- and $\tilde{\gamma}^0_-$ to $\mathbf{Cpt}^{n-1}_{k-1}$. Therefore, the category $\mathrm{Comp}^1(\mathbf{c}_n)^{op}$ is filtered. For the second statement, we only need to apply [15, 5.3.1.13, 5.3.1.18]. □

The following can be regarded as a higher categorical generalization of [3, 3.3.2].

Corollary 4.7. *Let $(\mathcal{C},\mathcal{E}_1,\mathcal{E}_2)$ be a 2-marked category satisfying the conditions of 4.6. Then the map*

$$\delta_2^*\mathrm{N}(\mathcal{C})_{\mathcal{E}_1,\mathcal{E}_2}\to\mathrm{N}(\mathcal{C})$$

is a categorical equivalence.

Proof. This follows immediately from 4.4 and 4.6. □

5. Combinatorics of finite lattices

In this section, we prove some results on the combinatorics of finite lattices that will be used in the next section.

By a *lattice* we mean a partially ordered set admitting products and coproducts of two elements. Equivalently, a lattice is a partially ordered set admitting finite nonempty products and coproducts. We write $\min\{p_1,\dots,p_n\}$ (resp. $\max\{p_1,\dots,p_n\}$) for the product (resp. coproduct) of finitely many elements $\{p_1,\dots,p_n\}$ $(n\ge1)$ of a lattice P. Morphisms of lattices are maps of the underlying partially ordered sets preserving finite nonempty products and coproducts. We say that a morphism of lattices is *injective* (resp. *surjective*) if the underlying map is injective (resp. surjective). A sublattice Q of P is a subset stable under finite nonempty products and coproducts. An injective morphism of lattices $i\colon Q\to P$ can be identified with the sublattice inclusion $i(Q)\subseteq P$. An interval sublattice of a lattice P is a sublattice of the form $P_{p//p'}$, where $p,p'\in P$. A lattice P is *finite* if its underlying set is finite. In particular, a finite lattice admits arbitrary products and coproducts and we will often write $-\infty_P$ (resp. $+\infty_P$) for the initial (resp. final) object of P. If Q is a sublattice of a finite lattice P and if $p\le p'\le p''$ in P with $p,p''\in Q$ implies that $p'\in Q$, then Q is an interval sublattice of P. We study some combinatorics of finite lattices.

Definition 5.1. Let P be a finite lattice. A (finite) collection $B=\{p_1,\dots,p_d\}$ of elements of $P-\{-\infty_P\}$ is called a *basis* of P if the following map

$$\prod_{i=1}^d P_{-\infty_P//p_i}\to P;\quad (q_1,\dots,q_d)\mapsto\max\{q_1,\dots,q_d\}$$

is an isomorphism (of lattices). We call B a *refinement* of another basis $B'=\{p'_1,\dots,p'_e\}$ if there exists a (necessarily unique surjective) function $f\colon B\to B'$ such that $p'_j=\max f^{-1}(p'_j)$ for $j=1,\dots,e$, and

we will write $B \preceq B'$. The set of all bases of P with $\preceq$ forms a partially ordered set $\mathfrak{B}_P^-$. We define the partially ordered set $\mathfrak{B}_P^+$ of *cobases* of P to be $\mathfrak{B}_{P^{op}}^-$.

The map f in the preceding definition can be characterized as sending $p \in B$ to the unique element $p' \in B'$ such that $p \leq p'$. There is a canonical isomorphism $\mathfrak{B}_P^- \simeq \mathfrak{B}_P^+$ sending the basis $B = \{p_1, \dots, p_d\}$ to the cobasis $B^\vee = \{\max(B - \{p_1\}), \dots, \max(B - \{p_d\})\}$. We will write $\mathfrak{B}_P = \mathfrak{B}_P^+ \simeq \mathfrak{B}_P^-$ and we have the following.

Lemma 5.2. *The partially ordered set $\mathfrak{B}_P$ is finite, and admits products and the initial object.*

The product of $B_1, \dots, B_k$ will be denoted by $\operatorname{refn}\{B_1, \dots, B_k\}$. The initial object of $\mathfrak{B}_P$ will be called the *maximal basis* of P whose cardinality will be called the *dimension* of P and denoted by $\dim P$.[4] In particular, when P consists of only one element, $\dim P = 0$.

Proof. Since $\mathfrak{B}_P$ is a subset of $\{0,1\}^P$, it is finite. The case that P consists of only one element is trivial and we will assume that P has more than one elements.

We now prove that products exist in $\mathfrak{B}_P$. Let $B_i = \{p_{i,1}, ..., p_{i,d_i}\}$ be elements of $\mathfrak{B}_P$ with $i = 1, \dots, k$ and $k > 0$. Let B be the set of elements of the form $\min\{p_{1,j_1}, \dots, p_{k,j_k}\}$ with $1 \leq j_i \leq d_i$, if it is not $-\infty_P$. Then B is a basis. We claim that B is the product of $B_1, \dots, B_k$. Without lost of generality, we show that $B \preceq B_1$. Let f be the function sending $\min\{p_{1,j_1}, \dots, p_{k,j_k}\} \neq -\infty_P$ to p_{1,j_1}, which is well-defined since $\min\{p_{1,j_1}, \dots, p_{k,j_k}\} = \min\{p_{1,j_1'}, \dots, p_{k,j_k'}\}$ implies that $p_{1,j_1} = p_{1,j_1'}$. Moreover, we have $p_{1,j_1} = \max f^{-1}(p_{1,j_1})$. Let B' be another basis of P such that $B' \preceq B_i$ for $i = 1, \dots, k$ with functions $f_i \colon B' \to B_i$ as in the definition. We define $f \colon B' \to B$ as $f(p') = \min\{f_1(p'), \dots, f_k(p')\}$, which is greater than or equal to p' and hence not $-\infty_P$. It is easy to check that f exhibits B' as a refinement of B. Therefore, nonempty products exist in $\mathfrak{B}_P$. Moreover, $\mathfrak{B}_P$ has the obvious final object $\{+\infty_P\}$.

Finally, the product of all elements of $\mathfrak{B}_P$ gives the initial object. $\square$

Given an interval sublattice inclusion $i \colon Q \to P$ of a finite lattice P, we construct a *restriction* functor $i^\dagger \colon \mathfrak{B}_P \to \mathfrak{B}_Q$ as follows. Given a basis $B_P = \{p_1, \dots, p_d\}$ of P, let $i^\dagger p_i = \max\{-\infty_Q, \min\{+\infty_Q, p_i\}\}$ which is in Q. The set $i^\dagger B_P := \{i^\dagger p_1, \dots, i^\dagger p_d\} \backslash \{-\infty_Q\}$ is a basis of Q. We have a map $i^\dagger B_P \to B_P$ sending $i^\dagger p_j$ to p_j whenever $i^\dagger p_j \neq -\infty_Q$. For another interval sublattice inclusion $j \colon R \to Q$, we have $j^\dagger \circ i^\dagger = (i \circ j)^\dagger$.

Lemma 5.3. *The map $i^\dagger$ is a functor,* i.e.*, an order-preserving map. Moreover, it preserves products.*

Proof. It suffices to show that $i^\dagger$ preserves finite products. It is clear that $i^\dagger\{+\infty_P\} = \{+\infty_Q\}$. Let $B_j = \{p_{j,1}, \dots, p_{j,d_j}\}$, $1 \leq j \leq k$ be elements of $\mathfrak{B}_P$ with $k > 0$. Then

$$\begin{aligned} i^\dagger \operatorname{refn}\{B_1, \dots, B_k\} &= \{\max\{-\infty_Q, \min\{+\infty_Q, p_{1,l_1}, \dots, p_{k,l_k}\}\} \mid 1 \leq l_j \leq d_j\} \backslash \{-\infty_Q\} \\ &= \{\min_{1 \leq j \leq k} \max\{-\infty_Q, \min\{+\infty_Q, p_{j,l_j}\}\} \mid 1 \leq l_j \leq d_j\} \backslash \{-\infty_Q\} = \operatorname{refn}\{i^\dagger B_1, \dots, i^\dagger B_k\}. \end{aligned}$$

Here for the second equality we have used the fact that, for every element p' of $\operatorname{refn}\{B_1, \dots, B_k\}$, the image of p_{j,l_j} under the projection from P to $P_{-\infty_P//p'}$ is either the initial object or the final object. $\square$

Given a sublattice inclusion $f \colon Q \to P$ of a finite lattice P such that f preserves initial and final objects, there is a *pullback* functor $f^\star \colon \mathfrak{B}_P \to \mathfrak{B}_Q$ defined in the following way. For every element B_P of $\mathfrak{B}_P$, let $(\mathfrak{B}_P)^f_{B_P/} \subseteq (\mathfrak{B}_P)_{B_P/}$ be the partially ordered subset consisting of bases which are subsets of $f(Q)$. Since $B, B' \in (\mathfrak{B}_P)^f_{B_P/}$ implies that $\operatorname{refn}\{B, B'\} \in (\mathfrak{B}_P)^f_{B_P/}$, $(\mathfrak{B}_P)^f_{B_P/}$ has a unique initial object, which we denote by $f^\star(B_P)$. It is easy to check that $f^\star(B_P)$ is a basis of Q. For another sublattice inclusion $g \colon R \to Q$ preserving initial and final objects, we have $g^\star \circ f^\star = (f \circ g)^\star$.

Lemma 5.4. *The map $f^\star$ is a functor,* i.e.*, an order-preserving map. In particular, $f^\star(\operatorname{refn}\{B_1, \dots, B_n\}) \preceq \operatorname{refn}\{f^\star(B_1), \dots, f^\star(B_n)\}$ for arbitrary objects $B_1, \dots, B_n$ of $\mathfrak{B}_P$.*

[4] This is not to be confused with the order dimension of P, which is always $\geq \dim P$.

Proof. Let $B \preceq B'$ be objects of $\mathfrak{B}_P$. By definition, we have $(\mathfrak{B}_P)^f_{B/} \supseteq (\mathfrak{B}_P)^f_{B'/}$. Thus $f^\star(B) \preceq f^\star(B')$, which proves the first assertion. The second assertion follows from the first. □

Lemma 5.5. *Consider a commutative diagram of finite lattices of the form*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "i"'] & P' \arrow[d, "j"] \\ Q \arrow[r, "g"] & Q' \end{tikzcd}$$

where i, j are interval sublattice inclusions, f, g are sublattice inclusions preserving initial and final objects. Then we have $f^\star \circ j^\dagger \preceq i^\dagger \circ g^\star$.

Proof. Let B be a basis of Q. By construction, $g^\star(B)$ is a basis of Q satisfying $B \preceq g^\star(B)$. By Lemma 5.3, $j^\dagger(B) \preceq j^\dagger(g^\star(B)) = i^\dagger(g^\star(B))$. Thus $i^\dagger(g^\star(B))$ belongs to $(\mathfrak{B}_P)^f_{j^\dagger(B)/}$. It follows that $f^\star(j^\dagger(B)) \preceq i^\dagger(g^\star(B))$. □

Suppose that $\pi\colon Q \to P$ is a surjective morphism of finite lattices. There is a *pushforward* functor $\pi_\star\colon \mathfrak{B}_Q \to \mathfrak{B}_P$ by $\pi_\star(B_Q) = \pi(B_Q)\backslash\{-\infty_P\}$. We have a map $B_P \to f^\star(B_P)$ sending $p \in B_P$ to the unique element $q \in f^\star(B_P)$ such that $p \le f(q)$. We have a map $\pi_\star(B_Q) \to B_Q$ sending $p \in \pi_\star(B_Q)$ to the unique element $q \in B_Q$ such that $p = \pi(q)$. For another surjective morphism $\varpi\colon R \to Q$ of finite lattices, we have $\pi_\star \circ \varpi_\star = (\pi \circ \varpi)_\star$.

Lemma 5.6. *The map $\pi_\star$ is a functor,* i.e.*, an order-preserving map. Moreover, it preserves products.*

Proof. It suffices to show that $\pi_\star$ preserves finite products, which is obvious. □

Lemma 5.7. *Consider a commutative diagram of finite lattices of the form*

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "i"'] & P' \arrow[d, "j"] \\ Q \arrow[r, "\varpi"] & Q' \end{tikzcd}$$

where i, j are interval sublattice inclusions, π, ϖ are surjective morphisms. Then we have $j^\dagger \circ \varpi_\star = \pi_\star \circ i^\dagger$.

Proof. For every object q of Q,

$$\begin{aligned} \max\{-\infty_{P'}, \min\{+\infty_{P'}, \varpi(q)\}\} =& \max\{\pi(-\infty_P), \min\{\pi(+\infty_P), \varpi(q)\}\} \\ =& \pi(\max\{-\infty_P, \min\{+\infty_P, q\}\}). \end{aligned}$$

Thus for every basis B of Q,

$$\begin{aligned} j^\dagger(\varpi_\star(B)) =& \{\max\{-\infty_{P'}, \min\{+\infty_{P'}, \varpi(q)\}\} \mid q \in B\}\backslash\{-\infty_{P'}\} \\ =& \{\pi(\max\{-\infty_P, \min\{+\infty_P, q\}\}) \mid q \in B\}\backslash\{-\infty_{P'}\} = \pi_\star(i^\dagger(B)). \end{aligned}$$

□

6. Cartesian gluing

In this section, we compare restricted nerves associated to different bimarkings. The main result is Theorem 6.11. Combining this with the results on multisimplicial descent in Section 4, we deduce Corollary 6.16, which includes Theorem 0.1 as a special case.

For every integer $n \ge 0$, we define a partially ordered set

$$\mathrm{Cart}^n = \{[(a_k, b_k)]_{0 \le k \le m} \mid a_0 \le \cdots \le a_m;\ b_0 \le \cdots \le b_m;\ 0 \le a_k, b_k \le n - k;\ m \ge 0\}.$$

We adopt the convention that $a_k = b_k = \infty$ if $k > m$. The partial order on Cart^n is given by $[(a_k, b_k)]_{0 \le k \le m} \le [(a'_k, b'_k)]_{0 \le k \le m'}$ if and only if $a_k \le a'_k$ and $b_k \le b'_k$ for all $k \ge 0$, or equivalently,

$m \geq m'$ and $a_k \leq a'_k$, $b_k \leq b'_k$ for all $0 \leq k \leq m'$. For example, Cart^1 is the partially ordered set generated by the following diagram

$$\begin{tikzcd} {[(0,0)^2]} \arrow[dr] & & \\ & {[(0,0)]} \arrow[r] \arrow[d] & {[(0,1)]} \arrow[d] \\ & {[(1,0)]} \arrow[r] & {[(1,1)]} \end{tikzcd}$$

For every face map $d^n_k \colon [n-1] \to [n]$, we have an induced map $\mathrm{Cart}(d^n_k) \colon \mathrm{Cart}^{n-1} \to \mathrm{Cart}^n$ which is given by

$$\begin{aligned} &\mathrm{Cart}(d^n_k)([(a_0,b_0)(a_1,b_1)\cdots]) \\ =&[(d^n_k(a_0), d^n_k(b_0))\cdots(d^n_k(a_{n-k-1}), d^n_k(b_{n-k-1}))(a_{n-k},b_{n-k})(a_{n-k},b_{n-k})(a_{n-k+1},b_{n-k+1})\cdots] \end{aligned}$$

where we put $d^n_k(\infty) = \infty$. For every degeneration map $s^n_k \colon [n+1] \to [n]$, we have an induced map $\mathrm{Cart}(s^n_k) \colon \mathrm{Cart}^{n+1} \to \mathrm{Cart}^n$ which is given by

$$\mathrm{Cart}(s^n_k)([(a_0,b_0)(a_1,b_1)\cdots]) = [(s^n_k(a_0), s^n_k(b_0))\cdots(s^n_k(a_{n-k}), s^n_k(b_{n-k}))(a_{n-k+2},a_{n-k+2})\cdots].$$

It is easy to check the compatibility conditions induced by those of d^n_k and s^n_k. In general, for a morphism $d \colon [m] \to [n]$ in $\mathbf{\Delta}$, we have the induced map $\mathrm{Cart}(d) \colon \mathrm{Cart}^m \to \mathrm{Cart}^n$.

We have a projection map $\pi^n \colon \mathrm{Cart}^n \to [n] \times [n]$ sending $[(a_k,b_k)]_{0\leq k\leq m}$ to (a_0,b_0), and the following diagrams commute:

$$\begin{tikzcd} \mathrm{Cart}^{n-1} \arrow[r, "\mathrm{Cart}(d^n_k)"] \arrow[d, "\pi^{n-1}"'] & \mathrm{Cart}^n \arrow[d, "\pi^n"] \\ {[n-1] \times [n-1]} \arrow[r, "d^n_k \times d^n_k"] & {[n] \times [n];} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Cart}^{n+1} \arrow[r, "\mathrm{Cart}(s^n_k)"] \arrow[d, "\pi^{n+1}"'] & \mathrm{Cart}^n \arrow[d, "\pi^n"] \\ {[n+1] \times [n+1]} \arrow[r, "s^n_k \times s^n_k"] & {[n] \times [n].} \end{tikzcd}$$

The map $\varsigma^n \colon [n]\times[n] \to \mathrm{Cart}^n$ sending (p,q) to $[(p,q)^{n+1-\max\{p,q\}}]$ is a morphism of lattices satisfying $\pi^n \circ \varsigma^n = \mathrm{id}_{[n]\times[n]}$. Moreover, the following diagrams commute:

$$\begin{tikzcd} {[n-1] \times [n-1]} \arrow[r, "d^n_k \times d^n_k"] \arrow[d, "\varsigma^{n-1}"'] & {[n] \times [n]} \arrow[d, "\varsigma^n"] \\ \mathrm{Cart}^{n-1} \arrow[r, "\mathrm{Cart}(d^n_k)"] & \mathrm{Cart}^n; \end{tikzcd} \qquad \begin{tikzcd} {[n+1] \times [n+1]} \arrow[r, "s^n_k \times s^n_k"] \arrow[d, "\varsigma^{n+1}"'] & {[n] \times [n]} \arrow[d, "\varsigma^n"] \\ \mathrm{Cart}^{n+1} \arrow[r, "\mathrm{Cart}(s^n_k)"] & \mathrm{Cart}^n. \end{tikzcd}$$

It is easy to check that Cart^n is a finite lattice; π^n, $\mathrm{Cart}(s^n_k)$ for $k = 0,\dots,n$ are surjective morphisms of finite lattices; ς^n, $\mathrm{Cart}(d^n_k)$ for $k = 0,\dots,n$ are sublattice inclusions.

We denote by $-(1) \colon \mathrm{Fun}([1], \mathrm{Cart}^n) \to \mathrm{Cart}^n$ the morphism of evaluation at 1. We construct a map (of sets) $\mathrm{crt}^n \colon [n]\times[n] \to \mathrm{Fun}([1],\mathrm{Cart}^n)$ such that the composed map $[n]\times[n] \xrightarrow{\mathrm{crt}^n} \mathrm{Fun}([1],\mathrm{Cart}^n) \xrightarrow{-(1)} \mathrm{Cart}^n$ is ς^n, as follows. For every $(p,q) \in [n]\times[n]$, we denote its image under crt^n by $\mathrm{crt}^n_{p,q}$ and we define

- $\mathrm{crt}^n_{p,p}$ to be $[(0,0)^{n+1-p}] \leq [(p,p)^{n+1-p}]$;
- $\mathrm{crt}^n_{p,q}$ to be $[(0,0)^{n+1-q}(p,0)^{q-p}] \leq [(p,q)^{n+1-q}]$ for $p < q$;
- $\mathrm{crt}^n_{p,q}$ to be $[(0,0)^{n+1-p}(0,q)^{p-q}] \leq [(p,q)^{n+1-p}]$ for $p > q$.

In particular,

- $\mathrm{crt}^n_{0,q}$ is $[(0,0)^{n+1}] \leq [(0,q)^{n+1-q}]$;
- $\mathrm{crt}^n_{p,0}$ is $[(0,0)^{n+1}] \leq [(p,0)^{n+1-p}]$.

One checks that for every $k=0,\dots,n$, we have the following commutative diagram:

$$\begin{tikzcd}
[n+1]\times[n+1] \arrow[r, "s_k^n\times s_k^n"] \arrow[d, "\mathrm{crt}^{n+1}"'] & {[n]\times[n]} \arrow[d, "\mathrm{crt}^n"] \\
\mathrm{Fun}([1],\mathrm{Cart}^{n+1}) \arrow[r, "\mathrm{Fun}([1],\mathrm{Cart}(s_k^n))"] & \mathrm{Fun}([1],\mathrm{Cart}^n)
\end{tikzcd}$$

and for every $(p,q)\in[n-1]\times[n-1]$, $\mathrm{crt}^n_{d_k^n(p),d_k^n(q)}(0)\le \mathrm{Cart}(d_k^n)(\mathrm{crt}^{n-1}_{p,q}(0))$.

For $(p,q)\in[n]\times[n]$, we let $\mathrm{Cart}^n_{p,q}=\mathrm{Cart}^n_{\mathrm{crt}^n_{p,q}(0)//\mathrm{crt}^n_{p,q}(1)}\subseteq\mathrm{Cart}^n$. We list the finest basis $B^n_{p,q}$ of $\mathrm{Cart}^n_{p,q}$:

- $B^n_{0,0}=\emptyset$;
- $B^n_{0,q}=\{[(0,q)^{n+1-q}]\}$ for $q>0$;
- $B^n_{p,0}=\{[(p,0)^{n+1-p}]\}$ for $p>0$;
- $B^n_{p,q}=\{[(p,0)^{n+1-p}],[(0,q)^{n+1-q}]\}$ for $p,q>0$.

The element $[(p,0)^{n+1-p}]$ (resp. $[(0,q)^{n+1-q}]$) is said to be *of type 1* (resp. *of type 2*).

For $n\ge 0$, let $I_n=\mathrm{Cpt}^n$ be the subset of $[n]\times[n]$ consisting of (p,q) with $0\le p\le q\le n$. For every element $c=(c(0)\le c(1))$ of $\mathrm{Fun}([1],\mathrm{Cart}^n)$, let $I_c\subseteq I_n$ be the subset consisting of (p,q) such that $\mathrm{crt}^n_{p,q}(0)\le c(0)\le c(1)\le \mathrm{crt}^n_{p,q}(1)$. Then $B^n_c=\mathrm{refn}\{i^\dagger_{c,p,q}B^n_{p,q}\}_{(p,q)\in I_c}$ is a basis of $\mathrm{Cart}^n_c:=\mathrm{Cart}^n_{c(0)//c(1)}$, where $i_{c,p,q}\colon \mathrm{Cart}^n_c\to\mathrm{Cart}^n_{p,q}$ is the induced interval sublattice inclusion. If $c'=(c'(0)\le c'(1))$ is another element of $\mathrm{Fun}([1],\mathrm{Cart}^n)$ such that $c'(0)\le c(0)\le c(1)\le c'(1)$, then there is a natural interval sublattice inclusion $i_{c,c'}\colon\mathrm{Cart}^n_c\to\mathrm{Cart}^n_{c'}$ and $I_{c'}\subseteq I_c$. Since $i^\dagger_{c,c'}$ preserves products, we have that $B^n_c\preceq i^\dagger_{c,c'}B^n_{c'}$. Then there is a map $\varphi_{c,c'}\colon B^n_c\to B^n_{c'}$ sending $\beta\in B^n_c$ to the unique element $\beta'\in B^n_{c'}$ such that $\beta\le i^\dagger_{c,c'}\beta'$. The inclusion $I_{c'}\subseteq I_c$ (resp. $I_c\subseteq I_n$) induces a (surjective) restriction map $\phi_{c,c'}\colon\{1,2\}^{I_c}\to\{1,2\}^{I_{c'}}$ (resp. $\phi_c\colon\{1,2\}^{I_n}\to\{1,2\}^{I_c}$) satisfying $\phi_{c,c'}\circ\phi_c=\phi_{c'}$.

We define a map $\iota\colon B^n_c\to\{1,2\}^{I_c}$ as follows. For every element (p,q) of I_c, since $B^n_c\preceq i^\dagger_{p,q}B^n_{p,q}$, we have a map $\varphi_{c,p,q}\colon B^n_c\to B^n_{p,q}$ sending $\beta\in B^n_c$ to the unique element $\beta'\in B^n_{p,q}$ such that $\beta\le i^\dagger_{p,q}\beta'$. We define the (p,q)-component of $\iota(\beta)$ to be the type of $\varphi_{c,p,q}(\beta)$.

Lemma 6.1. *Let c,c' be objects of $\mathrm{Fun}([1],\mathrm{Cart}^n)$ such that $c'(0)\le c(0)\le c(1)\le c'(1)$. Then the following diagram*

$$\begin{tikzcd}
B^n_c \arrow[r, "\iota"] \arrow[d, "\varphi_{c,c'}"'] & \{1,2\}^{I_c} \arrow[d, "\phi_{c,c'}"] \\
B^n_{c'} \arrow[r, "\iota"] & \{1,2\}^{I_{c'}}
\end{tikzcd}$$

commutes.

Proof. This follows from the obvious fact that, for every element (p,q) of $I_{c'}$, the diagram

$$\begin{tikzcd}
B^n_c \arrow[d, "\varphi_{c,c'}"'] \arrow[dr, "\varphi_{c,p,q}"] & \\
B^n_{c'} \arrow[r, "\varphi_{c',p,q}"'] & B^n_{p,q}
\end{tikzcd}$$

commutes. □

For every element $c = (c(0) \leq c(1))$ of $\mathrm{Fun}([1], \mathrm{Cart}^{n-1})$, let $c' = \mathrm{Fun}([1], \mathrm{Cart}(d^n_k))(c)$. For every $(p,q) \in I_c$, consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Cart}^{n-1}_c \arrow[r, "\mathrm{Cart}(d^n_k)|\mathrm{Cart}^{n-1}_c"] \arrow[d, "i_{c,p,q}"'] & \mathrm{Cart}^n_{c'} \arrow[r, "="] \arrow[d, "i'_{p,q}"'] & \mathrm{Cart}^n_{c'} \arrow[d, "i_{c',d^n_k(p),d^n_k(q)}"] \\
\mathrm{Cart}^{n-1}_{p,q} \arrow[r, "\mathrm{Cart}(d^n_k)"] & \mathrm{Cart}^n_{\mathrm{Cart}(d^n_k)(\mathrm{crt}^{n-1}_{p,q}(0))//\mathrm{Cart}(d^n_k)(\mathrm{crt}^{n-1}_{p,q}(1))} \arrow[r, "j_{p,q}"] & \mathrm{Cart}^n_{d^n_k(p),d^n_k(q)}
\end{tikzcd} \tag{6.1}$$

where $\mathrm{Cart}(d^n_k)$ and $\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_c$ are sublattice inclusions preserving initial and final objects, and the other arrows are interval sublattice inclusions. Therefore, we have

$$\begin{aligned}
(\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_c)^\star(i^\dagger_{c',d^n_k(p),d^n_k(q)} B^n_{d^n_k(p),d^n_k(q)}) &\preceq (\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_c)^\star(i'^\dagger_{p,q}(j^\dagger_{p,q} B^n_{d^n_k(p),d^n_k(q)})) \\
&= i^\dagger_{c,p,q}((\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_{p,q})^\star(j^\dagger_{p,q} B^n_{d^n_k(p),d^n_k(q)})) = i^\dagger_{c,p,q} B^{n-1}_{p,q},
\end{aligned}$$

which implies that

$$\begin{aligned}
(\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_c)^\star(B^n_{c'}) \preceq &(\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_c)^\star(\mathrm{refn}\{i^\dagger_{c',d^n_k(p),d^n_k(q)} B^n_{d^n_k(p),d^n_k(q)}\}_{(p,q)\in I_c}) \\
\preceq &\mathrm{refn}\{(\mathrm{Cart}(d^n_k) \mid \mathrm{Cart}^{n-1}_c)^\star(i^\dagger_{c',d^n_k(p),d^n_k(q)} B^n_{d^n_k(p),d^n_k(q)})\}_{(p,q)\in I_c} \\
\preceq &\mathrm{refn}\{i^\dagger_{c,p,q} B^{n-1}_{p,q}\}_{(p,q)\in I_c} = B^{n-1}_c.
\end{aligned}$$

Therefore, there is a map $\varphi(d^n_k)_c \colon B^n_{c'} \to B^{n-1}_c$ sending $\beta' \in B^n_{c'}$ to the unique element $\beta \in B^{n-1}_c$ such that $\beta' \leq \mathrm{Cart}(d^n_k)(\beta)$. The map d^n_k induces an injective map $I_c \to I_{c'}$ and hence a surjective map $\phi(d^n_k)_c \colon \{1,2\}^{I_{c'}} \to \{1,2\}^{I_c}$ by composition.

Lemma 6.2. *For every element $c = (c(0) \leq c(1))$ of $\mathrm{Fun}([1], \mathrm{Cart}^{n-1})$, let $c' = \mathrm{Fun}([1], \mathrm{Cart}(d^n_k))(c)$. The following diagram*

$$\begin{tikzcd}
B^n_{c'} \arrow[r, "\iota"] \arrow[d, "\varphi(d^n_k)_c"'] & \{1,2\}^{I_{c'}} \arrow[d, "\phi(d^n_k)_c"] \\
B^{n-1}_c \arrow[r, "\iota"] & \{1,2\}^{I_c}
\end{tikzcd}$$

commutes.

Proof. For $(p,q) \in I_c$, the map $\varphi_{c',d^n_k(p),d^n_k(q)}$ factors as $B^n_{c'} \xrightarrow{\varphi'_{p,q}} j^\dagger_{p,q} B^n_{d^n_k(p),d^n_k(q)} \to B^n_{d^n_k(p),d^n_k(q)}$, where $j_{p,q}$ appears in (6.1). The lemma follows from the following commutative diagram

$$\begin{tikzcd}
B^n_{c'} \arrow[r, "\varphi'_{p,q}"] \arrow[d, "\varphi(d^n_k)_c"'] & j^\dagger_{p,q} B^n_{d^n_k(p),d^n_k(q)} \arrow[d] \\
B^{n-1}_c \arrow[r, "\varphi_{c,p,q}"] & B^{n-1}_{p,q},
\end{tikzcd}$$

where the right vertical map is an isomorphism induced by $\mathrm{Cart}(d^n_k)$. □

For every element $c = (c(0) \leq c(1))$ of $\mathrm{Fun}([1], \mathrm{Cart}^{n+1})$, let $c' = \mathrm{Fun}([1], \mathrm{Cart}(s^n_k))(c)$. For every $(p,q) \in I_c$, consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Cart}^{n+1}_c \arrow[r, "\mathrm{Cart}(s^n_k)|\mathrm{Cart}^{n+1}_c"] \arrow[d, "i_{c,p,q}"'] & \mathrm{Cart}^n_{c'} \arrow[d, "i_{c',s^n_k(p),s^n_k(q)}"] \\
\mathrm{Cart}^{n+1}_{p,q} \arrow[r, "\mathrm{Cart}(s^n_k)"] & \mathrm{Cart}^n_{s^n_k(p),s^n_k(q)}
\end{tikzcd}$$

where $\mathrm{Cart}(s_k^n)$ and $\mathrm{Cart}(s_k^n) \mid \mathrm{Cart}_c^{n+1}$ are surjective morphisms. Therefore, we have

$$i^\dagger_{c',s_k^n(p),s_k^n(q)} B^n_{s_k^n(p),s_k^n(q)} = i^\dagger_{s_k^n(p),s_k^n(q)}((\mathrm{Cart}(s_k^n) \mid \mathrm{Cart}_c^{n+1})_\star(B^{n+1}_{p,q})) = \mathrm{Cart}(s_k^n)_\star(i^\dagger_{c,p,q} B^{n+1}_{p,q})$$

which implies that

$$\begin{aligned} B^n_{c'} =& \mathrm{refn}\{i^\dagger_{c',p,q} B^n_{p,q}\}_{(p,q)\in I_{c'}} = \mathrm{refn}\{i^\dagger_{s_k^n(p),s_k^n(q)} B^n_{s_k^n(p),s_k^n(q)}\}_{(p,q)\in I_c} \\ =& \mathrm{refn}\{(\mathrm{Cart}(s_k^n) \mid \mathrm{Cart}_c^{n+1})_\star(i^\dagger_{c,p,q} B^{n+1}_{p,q})\}_{(p,q)\in I_c} \\ =& (\mathrm{Cart}(s_k^n) \mid \mathrm{Cart}_c^{n+1})_\star(\mathrm{refn}\{i^\dagger_{c,p,q} B^{n+1}_{p,q}\}_{(p,q)\in I_c}) = (\mathrm{Cart}(s_k^n) \mid \mathrm{Cart}_c^{n+1})_\star(B^{n+1}_c). \end{aligned}$$

Therefore, there is a map $\varphi(s_k^n)_c \colon B^n_{c'} \to B^{n+1}_c$ sending $\beta' \in B^n_{c'}$ to the unique element $\beta \in B^{n+1}_c$ such that $\beta' = \mathrm{Cart}(s_k^n)(\beta)$. The map s_k^n induces a surjective map $I_c \to I_{c'}$ and hence an injective map $\phi(s_k^n)_c \colon \{1,2\}^{I_{c'}} \to \{1,2\}^{I_c}$ by composition.

Lemma 6.3. *For every element $c = (c(0) \le c(1))$ of $\mathrm{Fun}([1], \mathrm{Cart}^{n+1})$, let $c' = \mathrm{Fun}([1], \mathrm{Cart}(s_k^n))(c)$. The following diagram*

$$\begin{tikzcd} B^n_{c'} \arrow[r, "\iota"] \arrow[d, "\varphi(s_k^n)_c"'] & \{1,2\}^{I_{c'}} \arrow[d, "\phi(s_k^n)_c"] \\ B^{n+1}_c \arrow[r, "\iota"] & \{1,2\}^{I_c} \end{tikzcd}$$

commutes.

Proof. This follows from the obvious fact that, for every element (p,q) of I_c, the diagram

$$\begin{tikzcd} B^n_{c'} \arrow[r, "\varphi_{c',s_k^n(p),s_k^n(q)}"] \arrow[d, "\varphi(s_k^n)_c"'] & B^n_{s_k^n(p),s_k^n(q)} \arrow[d] \\ B^{n+1}_c \arrow[r, "\varphi_{c,p,q}"] & B^{n+1}_{p,q} \end{tikzcd}$$

commutes, where the right vertical map is induced by $\mathrm{Cart}(s_k^n)$. □

In general, for any morphism $d \colon [m] \to [n]$ in $\mathbf{\Delta}$, if c is an object of $\mathrm{Fun}([1], \mathrm{Cart}^m)$ and $c' = \mathrm{Fun}([1], \mathrm{Cart}(d))(c)$, then d induces a map $I_c \to I_{c'}$, which, in turn, induces a map $\phi(d)_c \colon \{1,2\}^{I_{c'}} \to \{1,2\}^{I_c}$.

Notation 6.4. We let $\mathcal{C}\mathrm{art}^n = \mathrm{N}(\mathrm{Cart}^n)$, $\mathcal{C}\mathrm{art}^n_c = \mathrm{N}(\mathrm{Cart}^n_c)$; $\mathcal{C}\mathrm{art}(d) = \mathrm{N}(\mathrm{Cart}(d))$ and $\mathcal{C}\mathrm{art}(d)_c = \mathcal{C}\mathrm{art}(d) \mid \mathcal{C}\mathrm{art}^n_c$ for a morphism d in $\mathbf{\Delta}$. We will still write π^n and ς^n for $\mathrm{N}(\pi^n)$ and $\mathrm{N}(\varsigma^n)$, respectively. We define

$$\boxplus^n = \bigcup_{0\le p\le q\le n} \mathrm{N}(\mathrm{Cart}^n_{p,q}) \subseteq \mathcal{C}\mathrm{art}^n.$$

The following lemma is crucial for our argument. The proof will be given in Lemma 7.5.

Lemma 6.5. *The inclusion $\boxplus^n \subseteq \mathcal{C}\mathrm{art}^n$ is inner anodyne.*

Let I, J be sets. For a map $\phi \colon \{1,2\} \coprod I \to \{1,2\} \coprod J$ with $\phi(\alpha) = \alpha$ for $\alpha = 1,2$, there is a natural map $\Lambda_\phi \colon (\Lambda^2_0)^J \to (\Lambda^2_0)^I$ which is the product of

- $(\Lambda^2_0)^J \to (\Lambda^2_0)^{\{\phi(i)\}} \xrightarrow{\sim} (\Lambda^2_0)^{\{i\}}$ if $\phi(i) \in J$;
- $(\Lambda^2_0)^J \to \Delta^{\{\alpha\}} \subseteq (\Lambda^2_0)^{\{i\}}$ if $\phi(i) = \alpha \in \{1,2\}$.

It is an anodyne inclusion if ϕ is surjective.

Let $\mathbb{I}_n = \{1,2\}^{I_n} \backslash (\{1\}^{I_n} \cup \{2\}^{I_n})$ (resp. $\mathbb{I}_c = \{1,2\}^{I_c} \backslash (\{1\}^{I_c} \cup \{2\}^{I_c})$ if $I_c \neq \emptyset$). Then $\{1,2\}^{I_n}$ (resp. $\{1,2\}^{I_c}$ if $I_c \neq \emptyset$) is canonically identified with $\{1,2\} \coprod \mathbb{I}_n$ (resp. $\{1,2\} \coprod \mathbb{I}_c$). We have induced maps $\phi_c \colon \mathbb{I}_n \to \mathbb{I}_c$, $\phi_{c,c'} \colon \mathbb{I}_c \to \mathbb{I}_{c'}$ and $\phi(d)_c \colon \mathbb{I}_{\mathrm{Fun}([1],\mathrm{Cart}(d))(c)} \to \mathbb{I}_c$ if $I_c \supseteq I_{c'} \neq \emptyset$. Applying the previous construction of Λ_ϕ to ϕ_c, we obtain an anodyne inclusion $\Lambda_{\phi_c} \colon (\Lambda^2_0)^{\mathbb{I}_c} \to (\Lambda^2_0)^{\mathbb{I}_n}$ for an object c of $\mathrm{Fun}([1], \mathrm{Cart}^n)$ such that $I_c \neq \emptyset$. We have the following lemma.

Lemma 6.6. *The inclusion*

$$\bigcup_{I_c \neq \emptyset} ((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat \subseteq ((\Lambda^2_0)^{\mathbb{I}_n})^\sharp \times (\boxplus^n)^\flat$$

is a trivial cofibration in $\mathcal{S}\mathrm{et}^+_\Delta$.

Proof. Choose an exhaustion of $\boxplus^n$ by a sequence of simplicial subsets

$$\emptyset = K^0 \subseteq K^1 \subseteq \cdots \subseteq K^m = \boxplus^n$$

such that each K^i, $1 \leq i \leq m$ is obtained from K^{i-1} by adjoining a single nondegenerate simplex $\sigma^i \colon \Delta^{l_i} \to K^{i+1}$. This induces inclusions

$$\bigcup_{I_c \neq \emptyset} ((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat = L^0 \subseteq L^1 \subseteq \cdots \subseteq L^m = ((\Lambda^2_0)^{\mathbb{I}_n})^\sharp \times (\boxplus^n)^\flat,$$

where $L^i = \bigcup_{I_c \neq \emptyset} ((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat \bigcup ((\Lambda^2_0)^{\mathbb{I}_n})^\sharp \times (K^i)^\flat$. Therefore, it suffices to show that the inclusion $L^{i-1} \subseteq L^i$ is a trivial cofibration in $\mathcal{S}\mathrm{et}^+_\Delta$ for all $1 \leq i \leq m$. However, this inclusion is a pushout of the map $(\Lambda_{\phi_c})^\sharp \times (\Delta^{d_i})^\flat \colon ((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\Delta^{d_i})^\flat \to ((\Lambda^2_0)^{\mathbb{I}_n})^\sharp \times (\Delta^{d_i})^\flat$, where $c \colon [1] \to \mathrm{Cart}^n$ is given by $c(0) = \sigma^i(0)$, $c(1) = \sigma^i(l_i)$. It then suffices to observe that since $(\Lambda_{\phi_c})^\sharp$ is a trivial cofibration in $\mathcal{S}\mathrm{et}^+_\Delta$, $(\Lambda_{\phi_c})^\sharp \times (\Delta^{l_i})^\flat$ is also a trivial cofibration by [15, 3.1.4.3]. $\square$

Notation 6.7. For a partially ordered set P admitting products of two elements, we let $\mu_P \colon P \times P \to P$ be the (order-preserving) map sending (p_1, p_2) to $\min\{p_1, p_2\}$. Let $\mathrm{diag}_P \colon P \to P \times P$ be the diagonal embedding, which is a section of μ_P. If P further admits coproducts of two elements and the initial object $-\infty$, we can define an (order-preserving) map

$$\tilde{\mu}_P \colon \mu_P^{-1}(-\infty) \times P \to P \times P$$

by sending (p_2, p_1) to $\max\{p_2, \mathrm{diag}_P(p_1)\}$ (which is the coproduct of p_2 and $\mathrm{diag}_P(p_1)$ in $P \times P$). The composite map $\mu_P \circ (\tilde{\mu}_P \mid \{p_2\} \times P) \colon \{p_2\} \times P \to P$ is the projection for every element $p_2 \in \mu_P^{-1}(-\infty)$.

For a set I, let

$$\mu^n_I = \mathrm{N}(\mu_{[n]^I}) \colon (\Delta^n)^I \times (\Delta^n)^I \to (\Delta^n)^I;$$
$$\tilde{\mu}^n_I = \mathrm{N}(\tilde{\mu}_{[n]^I}) \colon (\mu^n_I)^{-1}((\Delta^{\{0\}})^I) \times (\Delta^n)^I \to (\Delta^n)^I \times (\Delta^n)^I.$$

It is clear that $(\mu^1_I)^{-1}((\Delta^{\{0\}})^I) \simeq (\Lambda^2_0)^I$ canonically.

Assumption 6.8. Let $\mathcal{C}$ be an ∞-category admitting pullbacks, and K be a set. Let $\underline{\mathcal{E}} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}', \mathcal{E}_k\}_{k \in K}$ be a collection of sets of edges of $\mathcal{C}$ such that

- $\mathcal{E}' \subseteq \mathcal{E}_1 \cap \mathcal{E}_2$;
- $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}'$ are admissible (Definition 3.3);
- $\mathcal{E}_k$ $(k \in K)$ contains every degenerate edge and is stable under composition and pullback.

Definition 6.9. Let I be a set. A $(\{1,2\} \coprod I \coprod K)$-bimarked simplicial set $(\mathcal{C}, \mathcal{B})$ is $\underline{\mathcal{E}}$-*related* if

- $(\mathcal{C}, \mathcal{B}) \subseteq U(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}', \dots, \mathcal{E}', \{\mathcal{E}_k\}_{k \in K})$ where $\mathcal{E}'$ is repeated I times.
- Each set in $\mathcal{B}$ is stable under pullback, vertical composition and horizontal composition.
- $B_{ii} = \mathcal{E}_i *_{\mathcal{C}} \mathcal{E}_i$ for $i = 1, 2$.
- $\mathcal{B}_{12}$ is biadmissible (Definition 3.6) and contains $\mathcal{E}_1 *^{\mathrm{cart}}_{\mathcal{C}} \mathcal{E}_2$.
- $\mathcal{B}_{12} \cap (\mathcal{E}_1 \cap \mathcal{E}_2) *_{\mathcal{C}} (\mathcal{E}_1 \cap \mathcal{E}_2) = \mathcal{B}_{21} \cap (\mathcal{E}_1 \cap \mathcal{E}_2) *_{\mathcal{C}} (\mathcal{E}_1 \cap \mathcal{E}_2)$.
- $\mathcal{B}_{1k}$ and $\mathcal{B}_{2k}$ are vertically admissible for $k \in K$.
- $\mathcal{B}_{1k} \cap (\mathcal{E}_1 \cap \mathcal{E}_2) *_{\mathcal{C}} \mathcal{E}_k = \mathcal{B}_{2k} \cap (\mathcal{E}_1 \cap \mathcal{E}_2) *_{\mathcal{C}} \mathcal{E}_k$ for $k \in K$.

Note that if $I = \emptyset$, $\mathcal{E}'$ is irrelevant to the definition.

Lemma 6.10. *Let* $L \subseteq K$ *be a subset. Let* $(\mathcal{C}, \mathcal{B})$ *be an* $\underline{\mathcal{E}}$-*related* $(\{1,2\} \coprod K)$-*bimarked* ∞-*category,* $\mathcal{D}$ *be an* ∞-*category and* $f \colon \delta^*_{\{1,2\} \coprod K, L} \delta_*^{(\{1,2\} \coprod K)++}(\mathcal{C}, \mathcal{B}) \to \mathcal{D}$ *be a functor. If for every edge* g *in* $\mathcal{E}'$,

the square $g \circ \mu_1^1$ is in $\mathcal{B}_{12}$. Then for every set I and simplicial set S, there is a natural map

$$f_S^I\colon ((\Lambda_0^2)^I)^\sharp \times (\delta^*_{\{1,2\}\coprod I\coprod K,L}\delta_*^{(\{1,2\}\coprod I\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I)))^\flat \to \mathrm{Fun}(S,\mathcal{D})^\natural$$

(Notation 3.7), functorial in both I and S, where $(\mathcal{C},\mathcal{B}^I)$ is an $\underline{\mathcal{E}}$-related $(\{1,2\}\coprod I\coprod K)$-bimarked simplicial set with

- $\mathcal{B}^I_{jj'} = \mathcal{B}_{jj'}$ *if* $j,j' \in \{1,2\}\coprod K$;
- $\mathcal{B}^I_{ii} = \mathcal{E}_0 *_{\mathcal{C}} \mathcal{E}_0$ *for* $i \in I$;
- $\mathcal{B}^I_{1i} = \mathcal{B}_{12} \cap \mathcal{E}_1 *_{\mathcal{C}} \mathcal{E}'$ *for* $i \in I$;
- $\mathcal{B}^I_{2i} = \mathcal{B}_{21} \cap \mathcal{E}_2 *_{\mathcal{C}} \mathcal{E}'$ *for* $i \in I$;
- $\mathcal{B}^I_{ii'} = \mathcal{B}_{12} \cap \mathcal{E}' *_{\mathcal{C}} \mathcal{E}'$ *for* $i,i' \in I$, $i \neq i'$;
- $\mathcal{B}^I_{ik} = \mathcal{B}_{1k} \cap \mathcal{E}' *_{\mathcal{C}} \mathcal{E}_k$ *for* $i \in I$ *and* $k \in K$.

Precisely, the functoriality means the following.

(1) *For every map $\phi\colon \{1,2\}\coprod I \to \{1,2\}\coprod J$ with $\phi(\alpha)=\alpha$ for $\alpha = 1,2$, the following diagram:*

$$\begin{tikzcd}
((\Lambda_0^2)^I)^\sharp \times (\delta^*_{\{1,2\}\coprod I\coprod K,L}\delta_*^{(\{1,2\}\coprod I\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I)))^\flat \arrow[dr, "f_S^I"] & \\
((\Lambda_0^2)^J)^\sharp \times (\delta^*_{\{1,2\}\coprod I\coprod K,L}\delta_*^{(\{1,2\}\coprod I\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I)))^\flat \arrow[u, "\Lambda^\sharp_\phi\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mathrm{view}^\flat_\phi"'] & \mathrm{Fun}(S,\mathcal{D})^\natural \\
((\Lambda_0^2)^J)^\sharp \times (\delta^*_{\{1,2\}\coprod J\coprod K,L}\delta_*^{(\{1,2\}\coprod J\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^J)))^\flat \arrow[ur, "f_S^J"'] &
\end{tikzcd}$$

commutes, where the map

$$\mathrm{view}_\phi\colon \delta^*_{\{1,2\}\coprod I\coprod K,L}\delta_*^{\{1,2\}\coprod I\coprod K}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I)) \to \delta^*_{\{1,2\}\coprod J\coprod K,L}\delta_*^{\{1,2\}\coprod J\coprod K}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^J))$$

sends an n-cell of the source given by a map $\alpha\colon S\times(\Delta^n)^{\{1,2\}\coprod I}\times\Delta_L^{[n_k]_{k\in K}} \to \mathcal{C}$ ($n_k = n$) to the n-cell of the target given by the composite map

$$S\times(\Delta^n)^{\{1,2\}\coprod J}\times\Delta_L^{[n_k]_{k\in K}} \xrightarrow{\mathrm{id}\times(\Delta^n)^\phi\times\mathrm{id}} S\times(\Delta^n)^{\{1,2\}\coprod I}\times\Delta_L^{[n_k]_{k\in K}} \xrightarrow{\alpha} \mathcal{C}.$$

Here, the map $(\Delta^n)^\phi$ is the product of

$$(\Delta^n)^{\{1,2\}\coprod J} \to (\Delta^n)^{\{\phi(i)\}} \xrightarrow{\sim} (\Delta^n)^{\{i\}}$$

for $i \in \{1,2\}\coprod I$.

(2) *For a map $\tau\colon S\to T$ to another simplicial set T, the following diagram:*

$$\begin{tikzcd}
((\Lambda_0^2)^I)^\sharp \times (\delta^*_{\{1,2\}\coprod I\coprod K,L}\delta_*^{(\{1,2\}\coprod I\coprod K)++}\mathrm{Map}^\natural(T,(\mathcal{C},\mathcal{B}^I)))^\flat \arrow[r, "f_T^I"] \arrow[d] & \mathrm{Fun}(T,\mathcal{D})^\natural \arrow[d] \\
((\Lambda_0^2)^I)^\sharp \times \delta^*_{\{1,2\}\coprod I\coprod K,L}(\delta_*^{(\{1,2\}\coprod I\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I)))^\flat \arrow[r, "f_S^I"] & \mathrm{Fun}(S,\mathcal{D})^\natural
\end{tikzcd}$$

commutes, where the vertical maps are induced by τ.

Proof. For every n-cell σ of $\delta^*_{\{1,2\}\coprod I\coprod K,L}\delta_*^{(\{1,2\}\coprod I\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I))$, we are going to define a map $f_\sigma\colon ((\Lambda_0^2)^I)^\sharp\times(\Delta^n)^\flat \to \mathrm{Fun}(S,\mathcal{D})^\natural$ in a functorial way, meeting all the requirements in the proposition. We identify σ with a map

$$\sigma\colon S\times\Delta^n\times\Delta^n\times(\Delta^n)^I\times\Delta_L^{[n_k]_{k\in K}} \to \mathcal{C},$$

where $n_k = n$. Therefore,

$$\sigma \circ (\mathrm{id} \times \mathrm{id} \times \mathrm{id} \times \mu_I^n \times \mathrm{id})\colon S \times \Delta^n \times \Delta^n \times (\Delta^n)^I \times (\Delta^n)^I \times \Delta_L^{[n_k]_{k\in K}} \to \mathcal{C} \tag{6.2}$$

determines an n-cell σ' of $\delta^*_{\{1,2\}\coprod(I\times\{1,2\})\coprod K,L}\delta_*^{(\{1,2\}\coprod(I\times\{1,2\})\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^{I\times\{1,2\}}))$. Since

$$\delta^*_{\{1,2\}\coprod(I\times\{1,2\})\coprod K,L} = \delta^*_{(I\times\{1,2\})\coprod\{*\}} \circ (\mathbf{\Delta}^f)^* \circ \mathrm{op}_L^K,$$

σ' induces a map

$$(\Delta^n)^I \times (\Delta^n)^I \times \Delta^n \to \delta^*_{\{1,2\}\coprod(I\times\{1,2\})\coprod K,L}\delta_*^{(\{1,2\}\coprod(I\times\{1,2\})\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B}^I)). \tag{6.3}$$

Let $\phi_I\colon \{1,2\}\coprod(I\times\{1,2\}) \to \{1,2\}$ be the map such that $\phi_I(\alpha)=\alpha$, $\phi_I((i,\alpha))=\alpha$ for $\alpha = 1,2$ and $i\in I$. Composing view_{ϕ_I} with (6.3), we obtain a map

$$(\Delta^n)^I \times (\Delta^n)^I \times \Delta^n \to \delta^*_{\{1,2\}\coprod K,L}\delta_*^{(\{1,2\}\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B})).$$

Composing with the map

$$(\mu_I^n)^{-1}((\Delta^{\{0\}})^I) \times \Delta^n \xrightarrow{\mathrm{diag}} (\mu_I^n)^{-1}((\Delta^{\{0\}})^I) \times (\Delta^n)^I \times \Delta^n \xrightarrow{\tilde{\mu}_n^I\times\mathrm{id}} (\Delta^n)^I \times (\Delta^n)^I \times \Delta^n,$$

we obtain a map

$$\sigma''\colon (\mu_I^n)^{-1}((\Delta^{\{0\}})^I) \times \Delta^n \to \delta^*_{\{1,2\}\coprod K,L}\delta_*^{(\{1,2\}\coprod K)++}\mathrm{Map}^\natural(S,(\mathcal{C},\mathcal{B})).$$

Since there is a canonical map from $(\Lambda_0^2)^I$ to $(\mu_I^n)^{-1}((\Delta^{\{0\}})^I)$ induced by the map $[1]\to[n]$ sending 0 to 0 and 1 to n, the map $f\circ(\sigma'' \mid (\Lambda_0^2)^I\times\Delta^n)$ will provide f_σ. It is easy to see that the whole construction is functorial in σ, I and S. Therefore, the proposition follows. $\square$

Let $(\mathcal{C},\mathcal{B})$ be an $\underline{\mathcal{E}}$-related $(\{1,2\}\coprod K)$-bimarked ∞-category. Let σ be an n-cell of $\delta^*_{\{1,2\}\coprod K,L}\delta_*^{(\{1,2\}\coprod K)++}(\mathcal{C},\mathcal{B})$ which corresponds to a map $\sigma\colon \Delta^n\times\Delta^n\times\Delta_L^{[n_k]_{k\in K}} \to \mathcal{C}$ ($n_k = n$). Let $\mathcal{K}\mathrm{art}(\sigma) = \mathcal{K}\mathrm{art}_{\mathcal{C},\mathcal{B}}(\sigma)$ be the fiber product of

$$\begin{tikzcd} & \mathrm{Fun}(\mathcal{C}\mathrm{art}^n\times\Delta_L^{[n_k]_{k\in K}},\mathcal{C})_{\mathrm{RKE}} \arrow[d, "\mathrm{res}"] \\ \{\sigma\} \arrow[r, hook] & \mathrm{Fun}(\Delta^n\times\Delta^n\times\Delta_L^{[n_k]_{k\in K}},\mathcal{C}) \end{tikzcd}$$

where $\mathrm{Fun}(\mathcal{C}\mathrm{art}^n\times\Delta_L^{[n_k]_{k\in K}},\mathcal{C})_{\mathrm{RKE}} \subseteq \mathrm{Fun}(\mathcal{C}\mathrm{art}^n\times\Delta_L^{[n_k]_{k\in K}},\mathcal{C})$ is the full subcategory spanned by functors $F\colon \mathcal{C}\mathrm{art}^n\times\Delta_L^{[n_k]_{k\in K}}\to\mathcal{C}$ that are right Kan extensions of $F\mid\Delta^n\times\Delta^n\times\Delta_L^{[n_k]_{k\in K}}$ along the inclusion induced by $\varsigma^n\colon\Delta^n\times\Delta^n\hookrightarrow\mathcal{C}\mathrm{art}^n$. By [15, 4.3.2.15], res is a trivial fibration. Therefore, $\mathcal{K}\mathrm{art}(\sigma)$ is a contractible Kan complex. Moreover, for a map $d\colon\sigma\to\sigma'$, there is a restriction map $\mathcal{K}\mathrm{art}(d)\colon\mathcal{K}\mathrm{art}(\sigma')\to\mathcal{K}\mathrm{art}(\sigma)$.

Theorem 6.11 (Cartesian gluing). *Let $\mathcal{C}$, $L\subseteq K$, $\underline{\mathcal{E}}$ be as in Assumption 6.8, $(\mathcal{C},\mathcal{B})\subseteq(\mathcal{C},\mathcal{B}')$ be two $\underline{\mathcal{E}}$-related $(\{1,2\}\coprod K)$-bimarked ∞-categories such that $\mathcal{B}_{ij}=\mathcal{B}'_{ij}$ except when $(i,j)=(1,2)$ or $(2,1)$. Suppose that*

(1) *For every square*

$$\begin{tikzcd} w \arrow[r] \arrow[d] & z \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

in $\mathcal{B}'_{12}$, the induced arrow $w\to y\times_x z$ is in $\mathcal{E}'$.

(2) *For every arrow $g\colon y \to x$ in $\mathcal{E}'$, the square $g \circ \mu_1^1$, that is,*

$$\begin{tikzcd} y \arrow[r, "="] \arrow[d, "="'] & y \arrow[d, "g"] \\ y \arrow[r, "g"] & x \end{tikzcd}$$

is in $\mathcal{B}_{12}$.

Then the natural map $g\colon \delta^*_{\{1,2\}\coprod K,L}\delta_*^{(\{1,2\}\coprod K)++}(\mathcal{C},\mathcal{B}) \to \delta^*_{\{1,2\}\coprod K,L}\delta_*^{(\{1,2\}\coprod K)++}(\mathcal{C},\mathcal{B}')$ is a categorical equivalence.

Proof. By Lemma 4.3, it suffices to show that for every ∞-category $\mathcal{D}$, $l \geq 0$ and every diagram

$$\begin{tikzcd} Y = \delta^*_{\{1,2\}\coprod K,L}\delta_*^{\{1,2\}\coprod K}(\mathcal{C},\mathcal{B}) \arrow[r, "v"] \arrow[d, hook, "g"] & \mathrm{Fun}(\Delta^l,\mathcal{D}) \arrow[d, "p"] \\ Z = \delta^*_{\{1,2\}\coprod K,L}\delta_*^{\{1,2\}\coprod K}(\mathcal{C},\mathcal{B}') \arrow[r, "w"] \arrow[ur, dotted, "u"] & \mathrm{Fun}(\partial\Delta^l,\mathcal{D}), \end{tikzcd}$$

there is a map u as the dotted arrow such that $p \circ u = w$ and that $u \circ g$ and v are homotopic over $\mathrm{Fun}(\partial\Delta^l,\mathcal{D})$.

Let σ be an n-cell of Z given by a map

$$\sigma\colon \Delta^n \times \Delta^n \times \Delta_L^{[n_k]_{k\in K}} \to \mathcal{C}$$

where $n_k = n$. Let $\tau(\sigma)$ be an m-cell of $\mathcal{K}\mathrm{art}(\sigma)$ given by a map

$$\tau(\sigma)\colon \mathcal{C}\mathrm{art}^n \times \Delta_L^{[n_k]_{k\in K}} \to \mathrm{Map}^\flat((\Delta^m)^\sharp,\mathcal{C}^\natural).$$

Let c be an object of $\mathrm{Fun}([1],\mathrm{Cart}^n)$ such that $I_c \neq \emptyset$. We are going to construct a map

$$\tilde{\tau}(\sigma)_c\colon ((\Lambda_0^2)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}_c^n)^\flat \times (\Delta^n)^\flat \to \mathrm{Fun}(\Delta^m,\mathrm{Fun}(\Delta^l,\mathcal{D}))^\natural$$

in a functorial way. By assumption (1), $\tau(\sigma)$ induces a map

$$\boxtimes_{\theta\in\{1,2\}^{I_c}} \prod_{\substack{\beta\in B_c^n \\ \iota(\beta)=\theta}} \mathrm{N}(\mathrm{Cart}^n_{c(0)//\beta}) \boxtimes \Delta_L^{n_k|k\in K} \to \delta_*^{(\{1,2\}\coprod\mathbb{I}_c\coprod K)++}\mathrm{Map}^\natural(\Delta^m,(\mathcal{C},\mathcal{B}^{\mathbb{I}_c}))$$

of $(\{1,2\}\coprod\mathbb{I}_c\coprod K)$-simplicial sets, where we have identified $\{1,2\}^{I_c}$ with $\{1,2\}\coprod\mathbb{I}_c$. Combining directions in K, the previous map induces

$$\tau(\sigma)_c\colon \mathcal{C}\mathrm{art}_c^n \times \Delta^n \to \delta^*_{\{1,2\}\coprod\mathbb{I}_c\coprod K,L}\delta_*^{(\{1,2\}\coprod\mathbb{I}_c\coprod K)++}\mathrm{Map}^\natural(\Delta^m,(\mathcal{C},\mathcal{B}^{\mathbb{I}_c})).$$

By assumption (2), we can apply Lemma 6.10 to v, obtaining the map $\tilde{\tau}(\sigma)_c = v^{\mathbb{I}_c}_{\Delta^m} \circ \tau(\sigma)_c$ as claimed. By Lemma 6.12, the collection of maps $\tilde{\tau}(\sigma)_c$ induces a map

$$\tilde{\tau}(\sigma)\colon (\Delta^m)^\sharp \times \left(\bigcup_{I_c\neq\emptyset}((\Lambda_0^2)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}_c^n)^\flat\right) \times (\Delta^n)^\flat \times (\Delta^l)^\flat \to \mathcal{D}^\natural.$$

By Lemma 6.10, the map $\tilde{\tau}(\sigma)$ is functorial in $\tau(\sigma)$. Therefore, it defines a map

$$\tilde{\sigma}_1\colon \mathcal{K}\mathrm{art}(\sigma)^\sharp \times \left(\bigcup_{I_c\neq\emptyset}((\Lambda_0^2)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}_c^n)^\flat\right) \times (\Delta^n)^\flat \times (\Delta^l)^\flat \to \mathcal{D}^\natural.$$

By Lemmas 6.13 and 6.14, for every map $d\colon \sigma \to \sigma'$ from an n-cell to an n'-cell of Z, the following diagram

$$\begin{tikzcd}
\mathcal{K}\mathrm{art}(\sigma)^\sharp \times \Big(\bigcup_{I_c\neq\emptyset}((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat\Big) \times (\Delta^n)^\flat \times (\Delta^l)^\flat \arrow[dr, "\tilde{\sigma}_1"] & \\
\mathcal{K}\mathrm{art}(\sigma')^\sharp \times \Big(\bigcup_{I_c\neq\emptyset}((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat\Big) \times (\Delta^n)^\flat \times (\Delta^l)^\flat \arrow[u, "\mathcal{K}\mathrm{art}(d)^\sharp\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\left(\bigcup_{I_c\neq\emptyset}\Lambda^\sharp_{\phi(d)_c}\times\mathcal{C}\mathrm{art}(d)^\flat_c\right)\times d^\flat\times\mathrm{id}"'] & \mathcal{D}^\natural \\
\mathcal{K}\mathrm{art}(\sigma')^\sharp \times \Big(\bigcup_{I_{c'}\neq\emptyset}((\Lambda^2_0)^{\mathbb{I}_{c'}})^\sharp \times (\mathcal{C}\mathrm{art}^{n'}_{c'})^\flat\Big) \times (\Delta^{n'})^\flat \times (\Delta^l)^\flat \arrow[ur, "\tilde{\sigma}'_1"'] &
\end{tikzcd} \tag{6.4}$$

commutes. Moreover, for every n-cell σ, we denote by $\tilde{\sigma}_2$ the composite map

$$\mathcal{K}\mathrm{art}(\sigma)^\sharp \times \Big(((\Lambda^2_0)^{\mathbb{I}_n})^{\triangleleft,\sharp} \times (\mathcal{C}\mathrm{art}^n)^\flat\Big) \times (\Delta^n)^\flat \times (\partial\Delta^l)^\flat \to (\Delta^n)^\flat \times (\partial\Delta^l)^\flat \to \mathcal{D}^\natural,$$

where the first map is a projection and the second map is induced by w. By Lemma 6.10, the maps $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ can be amalgamated together to give a map

$$\tilde{\sigma}_3\colon \mathcal{K}\mathrm{art}(\sigma)^\sharp \times H \times (\Delta^n)^\flat \to \mathcal{D}^\natural,$$

where H equals

$$\left(\bigcup_{I_c\neq\emptyset}((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat\right) \times (\Delta^l)^\flat \coprod_{\left(\bigcup_{I_c\neq\emptyset}((\Lambda^2_0)^{\mathbb{I}_c})^\sharp\times(\mathcal{C}\mathrm{art}^n_c)^\flat\right)\times(\partial\Delta^l)^\flat} \Big(((\Lambda^2_0)^{\mathbb{I}_n})^{\triangleleft,\sharp} \times (\mathcal{C}\mathrm{art}^n)^\flat\Big) \times (\partial\Delta^l)^\flat$$

Let $\mathcal{N}(\sigma)$ be the fiber product of

$$\begin{tikzcd}
& \mathrm{Map}^\sharp(((\Lambda^2_0)^{\mathbb{I}_n})^{\triangleleft,\sharp} \times (\mathcal{C}\mathrm{art}^n)^\flat \times (\Delta^n)^\flat \times (\Delta^l)^\flat, \mathcal{D}^\natural) \arrow[d, "\mathrm{res}"] \\
\mathcal{K}\mathrm{art}(\sigma) \arrow[r] & \mathrm{Map}^\sharp(H \times (\Delta^n)^\flat, \mathcal{D}^\natural)
\end{tikzcd}$$

where the horizontal map is induced by $\tilde{\sigma}_3$. By Lemma 6.6, Lemma 6.5 and [15, 3.1.4.3], the inclusion

$$H \to ((\Lambda^2_0)^{\mathbb{I}_n})^{\triangleleft,\sharp} \times (\mathcal{C}\mathrm{art}^n)^\flat \times (\Delta^l)^\flat$$

is a trivial cofibration in $\mathcal{S}\mathrm{et}^+_\Delta$. Therefore, the map res is a trivial fibration. In particular, $\mathcal{N}(\sigma)$ is a contractible Kan complex. We let $\Phi(\sigma)$ be the composite map

$$\mathcal{N}(\sigma) \to \mathrm{Map}^\sharp(((\Lambda^2_0)^{\mathbb{I}_n})^{\triangleleft,\sharp} \times (\mathcal{C}\mathrm{art}^n)^\flat \times (\Delta^n)^\flat \times (\Delta^l)^\flat, \mathcal{D}^\natural) \to \mathrm{Fun}(\Delta^n, \mathrm{Fun}(\Delta^l, \mathcal{D}))$$

with image contained in $\mathrm{Map}^\sharp((\Delta^n)^\sharp, \mathrm{Fun}(\Delta^l,\mathcal{D})^\natural)$, where the second map is induced by the inclusion

$$\Delta^n \xrightarrow{\mathrm{diag}} (\Delta^n \times \Delta^n) \times \Delta^n \xrightarrow{\varsigma^n\times\mathrm{id}} \mathcal{C}\mathrm{art}^n \times \Delta^n \simeq \{-\infty\} \times \mathcal{C}\mathrm{art}^n \times \Delta^n \to ((\Lambda^2_0)^{\mathbb{I}_n})^\triangleleft \times \mathcal{C}\mathrm{art}^n \times \Delta^n.$$

By (6.4), the collection of $\Phi(\sigma)$ defines a morphism $\Phi\colon \mathcal{N} \to \mathrm{Map}[Z, \mathrm{Fun}(\Delta^l, \mathcal{D})]$ in $(\mathcal{S}\mathrm{et}_\Delta)^{(\mathbf{\Delta}_{/Z})^{op}}$. Moreover, the map v provides a section of $g^*\mathcal{N}$. Applying Lemma 2.4, we obtain the desired map u. □

Lemma 6.12. *Let c, c' be objects of $\mathrm{Fun}([1], \mathrm{Cart}^n)$ such that $c'(0) \leq c(0) \leq c(1) \leq c'(1)$ with $I_{c'} \neq \emptyset$. The following diagram*

$$\begin{tikzcd}
((\Lambda^2_0)^{\mathbb{I}_{c'}})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat \times (\Delta^n)^\flat \arrow[r, hook] \arrow[d, "\Lambda^\sharp_{\phi_{c,c'}}\times\mathrm{id}\times\mathrm{id}"'] & ((\Lambda^2_0)^{\mathbb{I}_{c'}})^\sharp \times (\mathcal{C}\mathrm{art}^n_{c'})^\flat \times (\Delta^n)^\flat \arrow[d, "\tilde{\tau}(\sigma)_{c'}"] \\
((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}^n_c)^\flat \times (\Delta^n)^\flat \arrow[r, "\tilde{\tau}(\sigma)_c"] & \mathrm{Fun}(\Delta^m, \mathrm{Fun}(\Delta^l, \mathcal{C}))^\natural
\end{tikzcd}$$

commutes.

Proof. By Lemma 6.10, we only need to prove that the following diagram

$$\begin{tikzcd}
\mathcal{C}\mathrm{art}_c^n \times \Delta^n \arrow[r, "\tau(\sigma)_c"] \arrow[d, hook] & \delta^*_{\{1,2\}\coprod \mathbb{I}_c \coprod K, L}\delta_*^{(\{1,2\}\coprod \mathbb{I}_c \coprod K)++}\mathrm{Map}^\natural(\Delta^m, (\mathcal{C}, \mathcal{B}^{\mathbb{I}_c})) \arrow[d, "\mathrm{view}_{\phi_{c,c'}}"] \\
\mathcal{C}\mathrm{art}_{c'}^n \times \Delta^n \arrow[r, "\tau(\sigma)_{c'}"] & \delta^*_{\{1,2\}\coprod \mathbb{I}_{c'} \coprod K, L}\delta_*^{(\{1,2\}\coprod \mathbb{I}_{c'} \coprod K)++}\mathrm{Map}^\natural(\Delta^m, (\mathcal{C}, \mathcal{B}^{\mathbb{I}_{c'}}))
\end{tikzcd}$$

commutes, which follows from Lemma 6.1. □

Lemma 6.13. *For every element $c = (c(0) \le c(1))$ of $\mathrm{Fun}([1], \mathrm{Cart}^{n-1})$ such that $I_c \neq \emptyset$, let $c' = \mathrm{Fun}([1], \mathrm{Cart}(d_k^n))(c)$. The following diagram*

$$\begin{tikzcd}
((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}_c^{n-1})^\flat \times (\Delta^{n-1})^\flat \arrow[d, "\Lambda^\sharp_{\phi(d_k^n)_c} \times \mathcal{C}\mathrm{art}(d_k^n)_c^\flat \times (d_k^n)^\flat"'] \arrow[dr, "\tilde{\tau}(\sigma)_c"] & \\
& \mathrm{Fun}(\Delta^m, \mathrm{Fun}(\Delta^l, \mathcal{C}))^\natural \\
((\Lambda^2_0)^{\mathbb{I}_{c'}})^\sharp \times (\mathcal{C}\mathrm{art}_{c'}^{n})^\flat \times (\Delta^{n})^\flat \arrow[ur, "\tilde{\tau'}(\sigma')_{c'}"] &
\end{tikzcd}$$

commutes, where $\sigma' = d_k^n(\sigma)$ and $\tau = \mathcal{K}\mathrm{art}(d_k^n)(\tau')$.

Proof. By Lemma 6.10, we only need to prove that the following diagram

$$\begin{tikzcd}
\mathcal{C}\mathrm{art}_c^{n-1} \times \Delta^{n-1} \arrow[r, "\tau(\sigma)_c"] \arrow[d, "\mathcal{C}\mathrm{art}(d_k^n)_c \times d_k^n"'] & \delta^*_{\{1,2\}\coprod \mathbb{I}_c \coprod K, L}\delta_*^{(\{1,2\}\coprod \mathbb{I}_c \coprod K)++}\mathrm{Map}^\natural(\Delta^m, (\mathcal{C}, \mathcal{B}^{\mathbb{I}_c})) \\
\mathcal{C}\mathrm{art}_{c'}^n \times \Delta^n \arrow[r, "\tau'(\sigma)_{c'}"] & \delta^*_{\{1,2\}\coprod \mathbb{I}_{c'} \coprod K, L}\delta_*^{(\{1,2\}\coprod \mathbb{I}_{c'} \coprod K)++}\mathrm{Map}^\natural(\Delta^m, (\mathcal{C}, \mathcal{B}^{\mathbb{I}_{c'}})) \arrow[u, "\mathrm{view}_{\phi(d_k^n)_c}"']
\end{tikzcd}$$

commutes, which follows from Lemma 6.2. □

Lemma 6.14. *For every element $c = (c(0) \le c(1))$ of $\mathrm{Fun}([1], \mathrm{Cart}^{n+1})$ such that $I_c \neq \emptyset$, let $c' = \mathrm{Fun}([1], \mathrm{Cart}(s_k^n))(c)$. The following diagram*

$$\begin{tikzcd}
((\Lambda^2_0)^{\mathbb{I}_c})^\sharp \times (\mathcal{C}\mathrm{art}_c^{n+1})^\flat \times (\Delta^{n+1})^\flat \arrow[d, "\Lambda^\sharp_{\phi(s_k^n)_c} \times \mathcal{C}\mathrm{art}(s_k^n)_c^\flat \times (s_k^n)^\flat"'] \arrow[dr, "\tilde{\tau}(\sigma)_c"] & \\
& \mathrm{Fun}(\Delta^m, \mathrm{Fun}(\Delta^l, \mathcal{C}))^\natural \\
((\Lambda^2_0)^{\mathbb{I}_{c'}})^\sharp \times (\mathcal{C}\mathrm{art}_{c'}^{n})^\flat \times (\Delta^{n})^\flat \arrow[ur, "\tilde{\tau'}(\sigma')_{c'}"] &
\end{tikzcd}$$

commutes, where $\sigma' = s_k^n(\sigma)$ and $\tau = \mathcal{K}\mathrm{art}(s_k^n)(\tau')$.

Proof. The proof is similar to the proof of Lemma 6.13. Lemma 6.2 in the last step of the proof is replaced by Lemma 6.3. □

Corollary 6.15. *Let $L \subseteq K$ be a subset, $\mathcal{C}$ be an ∞-category admitting pullbacks, and $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2, \{\mathcal{E}_k\}_{k\in K})$ be a $(\{1,2\}\coprod K)$-marked ∞-category such that $\mathcal{E}_1, \mathcal{E}_2$ are admissible and $\mathcal{E}_k$ is stable under composition and pullback for all $k \in K$. Suppose that there is a finite sequence*

$$\mathcal{E}'_0 \subseteq \mathcal{E}'_1 \subseteq \cdots \subseteq \mathcal{E}'_l = \mathcal{E}_1 \cap \mathcal{E}_2 \tag{6.5}$$

of admissible sets of edges of $\mathcal{C}$, where $\mathcal{E}'_0$ is the set of all equivalences, satisfying the following assumption

- *If an edge $y \to x$ is in $\mathcal{E}'_i$, then its diagonal $y \to y \times_x y$ is in $\mathcal{E}'_{i-1}$ for $i = 1, \ldots, l$.*

Then for every $\underline{\mathcal{E}}$-related $(\{1,2\}\coprod K)$-bimarked ∞-category $(\mathcal{C},\mathcal{B})$, where $\underline{\mathcal{E}} = \{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}'_0,\mathcal{E}_k\}_{k\in K}$, the natural map $\delta^*_{\{1,2\}\coprod K,L}\delta_*^{\{1,2\}\coprod K}(\mathcal{C},\mathcal{B}) \to \delta^*_{\{1,2\}\coprod K,L}\delta_*^{\{1,2\}\coprod K}(\mathcal{C},\tilde{\mathcal{B}})$ is a categorical equivalence, where $\mathcal{B}_{12} = \mathcal{E}_1 *^{\mathrm{cart}}_{\mathcal{C}} \mathcal{E}_2$, $\tilde{\mathcal{B}}_{12} = \mathcal{E}_1 *_{\mathcal{C}} \mathcal{E}_2$ and $\mathcal{B}_{ij} = \tilde{\mathcal{B}}_{ij}$ for $(i,j) \neq (1,2),(2,1)$.

Proof. For $i = 0,\dots,l$, let $(\mathcal{C},\mathcal{B}'_i)$ be the $\underline{\mathcal{E}}$-related $(\{1,2\}\coprod K)$ ∞-category between $(\mathcal{C},\mathcal{B})$ and $(\mathcal{C},\tilde{\mathcal{B}})$ such that $\mathcal{B}'_{12}$ consists of squares

$$\begin{tikzcd} w \arrow[r] \arrow[d] & z \arrow[d] \\ y \arrow[r] & x \end{tikzcd}$$

satisfying that $w \to y \times_x z$ is in $\mathcal{E}'_i$. Then we have

$$(\mathcal{C},\mathcal{B}) = (\mathcal{C},\mathcal{B}'_0) \subseteq (\mathcal{C},\mathcal{B}'_1) \subseteq \cdots \subseteq (\mathcal{C},\mathcal{B}'_l) = (\mathcal{C},\tilde{\mathcal{B}}).$$

For each $i = 1,\dots,l$, the pair $(\mathcal{C},\mathcal{B}'_{i-1}) \subseteq (\mathcal{C},\mathcal{B}'_i)$ satisfies the assumptions of Theorem 6.11 with $\mathcal{E}' = \mathcal{E}'_i$. Therefore, the corollary follows. □

Corollaries 4.5 and 6.15 imply the following corollary.

Corollary 6.16. *Let $\mathcal{C}$ be an ∞-category admitting pullbacks, K be a finite set, $(\mathcal{C},\mathcal{E}_0,\mathcal{E}_1,\mathcal{E}_2,\{\mathcal{E}_k\}_{k\in K})$ be a $(\{0,1,2\}\coprod K)$-marked ∞-category such that $\mathcal{E}_1,\mathcal{E}_2$ are admissible and $\mathcal{E}_k$ is stable under composition and pullback for $k=0$ and for $k\in K$. Let $\alpha \in \{1,2\}$ and $L \subseteq K$. Assume the following conditions:*

1. $\mathcal{E}_1 *_{\mathcal{C}} \mathcal{E}_2 \subseteq \mathrm{Hom}((\Delta^1\times\Delta^1)^\sharp,(\mathcal{C},\mathcal{E}_0))$.
2. *For every simplex σ of $\mathcal{C}_{\mathcal{E}_0} \subseteq \mathcal{C}$, $\mathcal{K}\mathrm{omp}^{\alpha}_{\mathcal{C},\mathcal{E}_1,\mathcal{E}_2}(\sigma)$ is weakly contractible.*
3. *There exists a finite sequence*
$$\mathcal{E}'_0 \subseteq \mathcal{E}'_1 \subseteq \cdots \subseteq \mathcal{E}'_l = \mathcal{E}_1\cap\mathcal{E}_2$$
of admissible sets of edges of $\mathcal{C}$, where $\mathcal{E}'_0$ is the set of all equivalences, such that for every $i = 1,\dots,l$ and every edge $y\to x$ in $\mathcal{E}'_i$, its diagonal $y \to y\times_x y$ is in $\mathcal{E}'_{i-1}$.

Then the natural map

$$g\colon \delta^*_{\{1,2\}\coprod K,L}\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_1,\mathcal{E}_2,\{\mathcal{E}_k\}_{k\in K}} \to \delta^*_{\{0\}\coprod K,L}\mathcal{C}^{\mathrm{cart}}_{\mathcal{E}_0,\{\mathcal{E}_k\}_{k\in K}}$$

(Notation 3.8) is a categorical equivalence.

Remark 6.17. Assumption (3) of Corollary 6.16 and the existence of the sequence (6.5) in the assumptions of Corollary 6.15 and are automatically satisfied if all morphisms in $\mathcal{E}_1\cap\mathcal{E}_2$ are n-truncated ([15, 5.5.6.8]) for some finite number n (this condition is satisfied if $\mathcal{C}$ is equivalent to an $(n+1)$-category). In fact, in this case, we can take $l = n+2$, and $\mathcal{E}'_i \subseteq \mathcal{E}_1\cap\mathcal{E}_2$ to be the subset of $(i-2)$-truncated morphisms for $0 \le i \le l = n+2$. The set $\mathcal{E}'_i$ is admissible since the set of all $(i-2)$-truncated morphisms is.

7. Some inner anodyne maps

In this section, we prove that certain inclusions of simplicial sets are inner anodyne. These facts were used in Sections 4 and 6.

Lemma 7.1. *Let $P\subseteq Q$ and $P\subseteq R$ be full inclusions of partially order sets, such that*

1. *P, R admit coproducts of two elements which are preserved by the inclusion.*
2. *$Q-P$ is finite.*
3. *$p\le q$ in Q with $p\in P$ implies that $q\in P$.*

Then

$$\mathrm{N}(Q)\cup\mathrm{N}(R) \subseteq \mathrm{N}\left(Q\coprod_P R\right)$$

is inner anodyne.

Proof. The case $P = Q$ is trivial. Suppose that $Q - P = \{q_1, \dots, q_n\}$ with $n \geq 1$, we prove by induction on n. When $n = 1$, we write $q = q_1$. Consider the following diagram which is a pushout by our assumption (3):

$$\begin{tikzcd}
\mathrm{N}(Q_{q/}) \bigcup \mathrm{N}(R_{q/}) \arrow[r] \arrow[d] & \mathrm{N}\left(\left(Q \coprod_P R\right)_{q/}\right) \arrow[d] \\
\mathrm{N}(Q) \bigcup \mathrm{N}(R) \arrow[r] & \mathrm{N}\left(Q \coprod_P R\right)
\end{tikzcd}$$

where all arrows are inclusions. Moreover, the upper arrow is isomorphic to the inclusion

$$\mathrm{N}(P_{q/})^{\triangleleft} \coprod_{\mathrm{N}(P_{q/})} \mathrm{N}(R_{q/}) \subseteq \mathrm{N}\left(\left(Q \coprod_P R\right)_{q/}\right)$$

which is inner anodyne by [15, 2.1.2.3, 4.1.1.3(4), 4.1.3.1], because our assumption (1) implies that $P_{q//r}$ is filtered for every $r \in R_{q/}$.

In general, without lost of generality, we assume that $q = q_n$ is a minimal element of $Q - P$. Consider inclusions $P' \subseteq Q$ and $P' \subseteq R'$, where

$$P' = Q - \{q_n\}; \quad R' = P' \coprod_P R.$$

Since $P'_{q//r}$ is filtered for every $r \in R'_{q/}$, the inclusion

$$\mathrm{N}(Q) \cup \mathrm{N}(R') \subseteq \mathrm{N}\left(Q \coprod_{P'} R'\right)$$

is inner anodyne by the same argument as before. By induction, $\mathrm{N}(P') \cup \mathrm{N}(R) \subseteq \mathrm{N}(R')$ is inner anodyne. It follows that $\mathrm{N}(Q) \cup \mathrm{N}(R) \subseteq \mathrm{N}(Q) \cup \mathrm{N}(R')$ is inner anodyne. Therefore, the lemma is proved. $\square$

Lemma 7.2. *Let C be a finite partially ordered set which admits the final object ∞. Consider the diagonal inclusion $[n] \subseteq \mathrm{Cpt}^n$. Then the inclusion*

$$(\Delta^n \times \mathrm{N}(C)) \cup (\mathcal{C}\mathrm{pt}^n \times \{\infty\}) \subseteq \mathrm{N}\left(([n] \times C) \coprod_{[n] \times \{\infty\}} (\mathrm{Cpt}^n \times \{\infty\})\right)$$

is inner anodyne.

Proof. We apply Lemma 7.1 to $P = [n] \times \{\infty\}$, $Q = [n] \times C$ and $R = \mathrm{Cpt}^n \times \{\infty\}$. The assumptions are satisfied immediately. $\square$

Lemma 7.3. *Let P be a finite lattice and $p_1, \dots, p_s; q_1, \dots, q_s$ be elements of P such that $p_i \leq q_1 \leq \dots \leq q_s$ for $1 \leq i \leq s$, then the inclusion*

$$\bigcup_{i=1}^{s} \mathrm{N}(P_{p_i//q_i}) \subseteq \bigcup_{i=1}^{s} \mathrm{N}(P_{p_i//q_s})$$

is inner anodyne, where the unions are taken in $\mathrm{N}(P)$.

Proof. We proceed by induction on s. The case $s = 0$ is trivial. For $s \geq 1$, consider the inclusions

$$\bigcup_{i=1}^{s} P_{p_i//q_i} \subseteq P_{p_1//q_1} \cup \left(\bigcup_{i=2}^{s} P_{p_i//q_s}\right) \subseteq \bigcup_{i=1}^{s} P_{p_i//q_s}. \tag{7.1}$$

The first inclusion is a pushout of $\bigcup_{i=2}^{s} P_{p_i//q_i} \subseteq \bigcup_{i=2}^{s} P_{p_i//q_s}$, which is inner anodyne by the induction hypothesis. Let

$$P' = P_{p_1//q_1} \cap \left(\bigcup_{i=2}^{s} P_{p_i//q_s} \right); \quad Q = P_{p_1//q_1}; \quad R = \bigcup_{i=2}^{s} P_{p_i//q_s}.$$

Then the triple (P', Q, R) satisfies the assumptions in Lemma 7.1. It follows that the second inclusion in (7.1) is also inner anodyne, since we have

$$Q \coprod_{P'} R = \bigcup_{i=1}^{s} P_{p_i//q_s}; \quad \mathrm{N} \left(\bigcup_{i=j}^{s} P_{p_i//q_s} \right) = \bigcup_{i=j}^{s} \mathrm{N}(P_{p_i//q_s}),\ j = 1, 2.$$

□

Now we prove Lemmas 4.1 and 6.5.

Lemma 7.4. *The inclusion $\square^n \subseteq \mathfrak{Cpt}^n$ is inner anodyne.*

Proof. Note that Cpt^n is a finite lattice. By Lemma 7.3, the inclusion

$$\square^n = \bigcup_{i=0}^{n} \mathrm{N}\left(\mathrm{Cpt}^n_{(0,i)//(i,n)}\right) \subseteq \bigcup_{i=0}^{n} \mathrm{N}\left(\mathrm{Cpt}^n_{(0,i)//(n,n)}\right) = \mathrm{N}(\mathrm{Cpt}^n) = \mathfrak{Cpt}^n$$

is inner anodyne. □

Lemma 7.5. *The inclusion $\boxplus^n \subseteq \mathfrak{Cart}^n$ is inner anodyne.*

Proof. Recall that $\mathfrak{Cart}^n = \mathrm{N}(\mathrm{Cart}^n)$ and $\boxplus^n = \bigcup_{0 \le p \le q \le n} \mathrm{N}(\mathrm{Cart}^n_{p,q})$. It suffices to prove that for every $0 \le k < n$, the inclusion

$$\boxplus \cup \mathrm{N}\left(\mathrm{Cart}^n_{[(0,0)^{n-k}]//[(n,n)]}\right) \subseteq \boxplus \cup \mathrm{N}\left(\mathrm{Cart}^n_{[(0,0)^{n+1-k}]//[(n,n)]}\right)$$

is inner anodyne. This inclusion is the composite map of a pushout of the inclusion

$$\left(\bigcup_{i=0}^{n-k-1} \mathrm{N}(\mathrm{Cart}^n_{k,n-i}) \right) \cup \mathrm{N}\left(\mathrm{Cart}^n_{[(0,0)^{n-k}]//[(n,n)]}\right) \subseteq \mathrm{N}\left(\mathrm{Cart}^n_{[(0,0)^{n-k}(k,0)]//[(n,n)]}\right) \tag{7.2}$$

and a pushout of the inclusion

$$\mathrm{N}(\mathrm{Cart}^n_{k,k}) \cup \mathrm{N}\left(\mathrm{Cart}^n_{[(0,0)^{n-k}(k,0)]//[(n,n)]}\right) \subseteq \mathrm{N}\left(\mathrm{Cart}^n_{[(0,0)^{n-k+1}]//[(n,n)]}\right). \tag{7.3}$$

Thus it suffices to observe that both (7.2) and (7.3) are inner anodyne by Lemma 7.3. □

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